Tracking of Discrete-Time Unmodeled Reference Signals in Robotic Manipulators using Output Regulation Theory and High-Gain Observers

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***Abstract*—In the field of control, output regulation theory ensures the tracking of reference signals generated by a known dynamic exogenous system, referred to as the exosystem. However, there are applications where the exosystem may be unknown. In this context, the present paper proposes a controller based on Francis output regulation theory, where a high-gain observer (HGO) is used to estimate the states of the unmodeled reference signals. This observer is structurally integrated into the discrete form of Francis' output regulation equations as the exosystem. The proposed approach assumes that discrete-time unmodeled reference signals are obtained by transforming a given trajectory from the workspace into joint space of a robotic manipulator. Consequently, the accuracy of motion in Cartesian space depends on the tracking error in each joint. As a case study, a two-degree-of-freedom robot is presented, and its dynamic model is used as the basis for constructing the controller. Finally, simulations are performed using a multibody model developed in Simulink-Simscape to demonstrate that the proposed controller satisfies the stability and asymptotic regulation criteria.**

Link to graphical and video abstracts, and to code: https://latamt.ieeer9.org/index.php/transactions/article/view/9931

***Index Terms*—Reference tracking, discrete-time unmodeled signals, robotic manipulators, output regulation theory, high-gain observer, multibody simulation.**

I. Introduction

THE output regulation problem addresses the control of a dynamic system such that its output follows an exogenous reference signal in steady state, preserving key properties such as stability, disturbance rejection, and a bounded error. In this context, Francis in [1]–[3] established the structural criteria for the synthesis of linear multivariable regulators when the reference and disturbances are generated by an exogenous dynamic system, also known as exosystem, giving rise to the internal model principle. Subsequently, Isidori in [4] extended Francis' theory to the nonlinear case, demonstrating the existence of a control law that solves the nonlinear output regulation problem. However, this formulation involves the challenge of solving a set of nonlinear partial differential equations called as Francis–Isidori–Byrnes (FIB) equations.

The traditional approach in output regulation theory for exact reference tracking relies on prior knowledge of the dynamic model of the exosystem, for instance, in [5], [6], the internal model principle is employed in reference tracking problems, where the exosystem is a virtual leader. However, depending on the application, the exosystem may be unknown, in this regard, [7]–[9] have employed adaptive internal model observers to address the output regulation problem in various phenomena described by partial differential equations, subject to disturbances originating from an unknown finite-dimensional exosystem. Zhengtao in [10], address the issue of global output regulation for uncertain linear systems subject to unknown disturbances generated by an also unknown linear exosystem, where such disturbances are compensated through adaptive control techniques. Similarly, in [11], [12], the output regulation problem is addressed in multi-agent systems under complete lack of knowledge about the exosystem. The solvability of the Francis equations, and consequently the existence of a regulator, depends on accurate knowledge of the system dynamics. When this information is unavailable, data-driven models have been proposed in [13], [14] to solve the regulator problem.

The complexity of synthesizing a controller with adequate performance may depend on the availability of measurements for each state of the plant. However, due to technical or even economic constraints, this condition may not always be fulfilled. In such cases, high-gain observers (HGO) represents an effective alternative for estimating the plant states, as well as for designing feedback controllers in nonlinear systems [15]. HGO is an attractive research topic due its powerful framework for state estimation, where a wide range of applications have been developed e.g. [16]–[19]. On the other hand, it is possible to establish necessary conditions for output regulation by integrating an HGO as a exosystem. In [20]–[23], it is shown that, in continuous time, a solution to the reference tracking problem exists even when the exosystem is not associated with any explicit model.

Particularly, robotic manipulator tasks are typically defined within the robot's workspace. For joint-based controllers, motion accuracy depends on the tracking error at each joint. As previous mentioned, one advantage of Francis output regulation theory is to guarantee asymptotic tracking when

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the plant and exosystem are known, as demonstrated in [24]–[26]. However, in the presence of uncertainties in the plant or exosystem, researchers have developed alternative strategies, which are based on state estimation [24], self-learning techniques [27] or adaptive methods [28].

Therefore, this paper proposes the design of a controller based on Francis output regulation theory and the integration of a HGO, aimed at tracking reference signals that lack a dynamic model or cannot be directly represented by an exosystem. It is assumed that the reference signals are of a discrete-time nature, as they can be measured or computed by digital devices. For the purposes of this study, each point of the trajectories in the robot's joint space are obtained from the inverse kinematics solution. This approach leads to the use of Francis equations in their discrete-time form.

The structure of this paper is as follows: Section II establishes the conditions to solve the output regulation problem for unmodeled discrete-time reference signals. Section III presents the continuous-time equations of motion of the robot, which are subsequently discretized for controller implementation. Finally, Sections IV and V show the performance of the proposed control scheme and the conclusions, respectively.

II. Problem Statement

A. Regulation Theory

Consider a plant represented by a linear multivariable system described by (1):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Pw(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where the system state matrix is $A \in \mathbb{R}^{n\times n}$, the input matrix is $B \in \mathbb{R}^{n\times m}$, the disturbance matrix is $P \in \mathbb{R}^{n\times r}$, the output matrix is $C \in \mathbb{R}^{p\times n}$, the state vector is $x(t) \in \mathbb{R}^{n}$, the input is $u(t) \in \mathbb{R}^{m}$ and the output is $y(t) \in \mathbb{R}^{p}$. Similarly, the existence of an exogenous state variables is considered, which are given by the dynamic system (2):

$$\begin{aligned} \dot{w}(t) &= Sw(t), \\ y_{ref}(t) &= Qw(t), \end{aligned} \tag{2}$$

here the matrix of exogenous states is $S \in \mathbb{R}^{r\times r}$, the exosystem matrix output is $Q \in \mathbb{R}^{p\times r}$ the vector of exogenous states is $w(t) \in \mathbb{R}^{r}$ and the reference output is $y_{ref} \in \mathbb{R}^{p}$.

According to the Francis output regulation theory [1], the tracking condition requires that $y(t) \to y_{ref}(t)$ when $t \to \infty$, while maintaining the internal stability of the closed-loop system under the control action:

$$u(t) = \underbrace{K\big(x(t) - \Pi w(t)\big)}_{stabilizer} + \underbrace{\Gamma w(t)}_{regulator} \tag{3}$$

As shown in (3) $\Pi \in \mathbb{R}^{n\times r}$ is the steady-state mapping from $w(t)$ to $x(t)$ and $\Gamma \in \mathbb{R}^{m\times r}$ is the steady-state feedforward mapping from $w(t)$ to $u(t)$. Additionally the control signal $u(t)$ incorporates both the plant state $x(t)$ and the exosystem state $w(t)$. It is assumed that $(A + BK)$ is Hurwitz, as well as Π and Γ are obtained from the solution of the Francis equations:

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P, \\ C\Pi &= Q. \end{aligned} \tag{4}$$

B. High-gain Observer

Consider a high-gain observer given by:

$$\begin{aligned} \dot{x}(t) &= A_h x(t) + B_h \phi\big(x(t)\big), \\ y(t) &= C_h x(t), \end{aligned} \tag{5}$$

where the state vector is $x(t) \in \mathbb{R}^{\rho}$, the output or measurement vector is $y \in \mathbb{R}$, and $\phi\big(x(t)\big)$ is a partially or completely unknown nonlinear function that is locally Lipschitz. Additionally, according to [29], the matrices A_h, B_h, and C_h are of the following form:

$$A_h = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{\rho\times\rho} ; \qquad B_h = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho} ;$$

$$C_h = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{\rho} .$$

Based on (5), the estimated state variable $\hat{x}$ is obtained as:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_h \hat{x}(t) + B_h \phi_0\big(\hat{x}(t)\big) + H \cdot \big(y(t) - \hat{y}(t)\big), \\ \hat{y}(t) &= C_h \hat{x}(t), \end{aligned} \tag{6}$$

where $\phi\big(\hat{x}(t)\big)$ is considered a nominal locally Lipschitz model; furthermore, the gain $H \in \mathbb{R}^{\rho}$ and is obtained from:

$$H = \begin{bmatrix} \alpha_1/\varepsilon & \alpha_2/\varepsilon^2 & \dots & \alpha_\rho/\varepsilon^\rho \end{bmatrix}^T . \tag{7}$$

It is worth mentioning that the coefficient ε is associated with the decay rate of the estimation error, and depending on its value, a more or less pronounced *peaking* phenomenon may occur. Consequently, it is recommended that the value of ε be sufficiently small to mitigate this effect [29]. Likewise, α_ρ represents the coefficients of the Hurwitz polynomial $s^\rho + \alpha_1 s^{\rho-1} + \alpha_2 s^{\rho-2} + \cdots + \alpha_{\rho-1}s + \alpha_\rho$.

III. High-Gain Observer as Exosystem for the Generation of Unmodeled Signals

The observer must simultaneously perform two functions: (a) estimate the reference signal, and (b) generate the exosystem states to satisfy the regulation condition. Under the assumption that the reference signal is unknown, the nominal model function is considered to be $\phi_0\big(\hat{a}(t)\big) = 0$. In this case, the High-Gain observer (6) for the tracking of a reference signal $\psi(t)$ takes the following form [23]:

$$\dot{w}(t) = A_h w(t) - H \begin{bmatrix} 1 & 0 & ... & 0 \end{bmatrix} w(t) + H\psi(t) \tag{8}$$

For n reference signals, matrix S is a block-diagonal matrix containing the corresponding to each exogenous input matrices $S_i = (A_{h,i} - H_i\,[1\ 0 ... 0])w(t)$. S_H is also a block matrix whose elements H_i are obtained from (7), thus the exosistem integrating the HGO is:

$$\dot{w}(t) = Sw(t) + S_H \Psi(t) \tag{9}$$

Let $\Psi(t)$ denote the reference signals, whose model is unknown. As previously mentioned, the main challenge lies in the lack of a mathematical model that accurately describes the reference trajectory. However, these trajectories or reference signals can be measured or computed digitally, and therefore have a discrete-time nature. Consequently, it is appropriate to express (1) and (9) in their discrete form, i.e.:

$$\begin{aligned} x_{k+1} &= A_d x_k + B_d u_k \\ y_k &= C x_k \\ w_{k+1} &= S_d w_k + S_{Hd}\Psi_k \\ y_{ref,k} &= Q_{ref} w_k \end{aligned} \tag{10}$$

The matrices A_d, B_d, S_d, and S_{Hd} result from the Euler discretization, i.e.:

$$A_d = I + T_s A, \tag{11}$$
$$B_d = T_s B, \tag{12}$$
$$S_d = I + T_s S, \tag{13}$$
$$S_{Hd} = T_s S_H. \tag{14}$$
$$u_k = -K(x_k - \Pi w_k) + \Gamma w_k \tag{15}$$

The existence of the output regulation condition is subjected to: $e_k = x_k - \Pi w_k$ and $e_{k+1} = x_{k+1} - \Pi w_{k+1}$, then substituting x_{k+1} and w_{k+1}:

$$e_{k+1} = (A_d - B_d K)e_k + A_d\Pi + B_d\Gamma - \Pi(S_d - S_{Hd}Q_{ref}) \tag{16}$$

If $e_k \to 0$, then (16) is reduced to the discrete-time Francis equations, which ensure output regulation for unmodeled reference signals:

$$\begin{aligned} \Pi(S_d + S_{Hd}Q_{ref}) &= A_d\Pi + B_d\Gamma + P \\ C\Pi &= Q_{ref} \end{aligned} \tag{17}$$

The solution of elements Π and Γ will ensure the reference tracking under control law given in (15). It is important to note that, in the discrete-time setting, the stabilizing gain K must be chosen such that the eigenvalues of the closed-loop pair (A_d, B_b) lie strictly within the unit circle.

IV. Controller Implementation for Tracking Unmodeled Signals in Robotic Manipulators

Consider a robotic a 2 dof manipulator as shown in Fig. 1, with $l_1 = l_2 = 0.5$ m, $m_1 = 0.4$ kg, and $m_2 = 0.3$ kg, the variables θ_1 and θ_2 represent the angular positions of each link, while $\dot{\theta}_1$, $\dot{\theta}_2$ denote their respective derivatives. Therefore, the state vector is defined as:

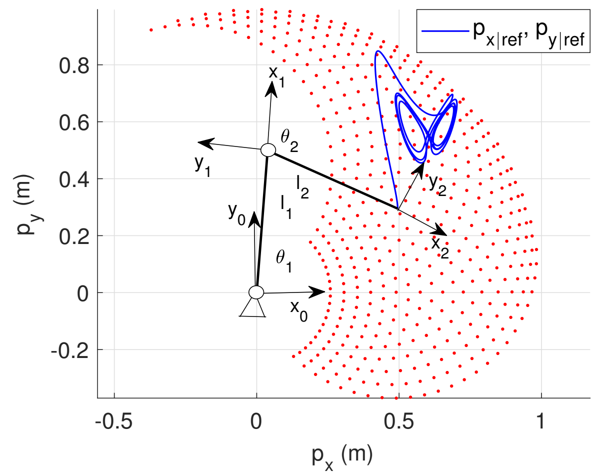


Fig. 1. Reference signal in the robotic manipulator workspace.

$$\begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} \Longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ f_1\big(x(t), u(t)\big) \\ x_4 \\ f_2\big(x(t), u(t)\big) \end{bmatrix}, \tag{18}$$

the functions $f_1\big(x(t), u(t)\big)$ and $f_2\big(x(t), u(t)\big)$ are obtained from the dynamic model using the Euler–Lagrange formulation [30]:

$$\begin{bmatrix} f_1\big(x(t), u(t)\big) \\ f_2\big(x(t), u(t)\big) \end{bmatrix} = \big[M\big(x(t)\big)\big]^{-1}\big[u(t) - c\big(x(t), \dot{x}(t)\big) - g\big(x(t)\big)\big] \tag{19}$$

where $M\big(x(t)\big)$ is the inertia matrix, $c\big(x(t), \dot{x}(t)\big)$ are the Coriolis and centrifugal terms, $g\big(x(t)\big)$ are the torques due to gravity acting on each joint, and $u(t)$ is the vector of generalized forces. The implementation of the controller is carry out by linearization of nonlinear system (19) around the operating points $[\bar{x}_1 \quad \bar{x}_2]^T = [85° \quad -109°]^T$, i.e.:

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=\bar{x}, u=\bar{u}}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=\bar{x}, u=\bar{u}},$$
$$C = \left.\frac{\partial h(x,u)}{\partial x}\right|_{x=\bar{x}, u=\bar{u}}.$$

Considering equations (11) and (12) with a sampling period of $T_S = 500$ μs, the discrete-time matrices A_d, B_d are given by:

$$A_d = \begin{bmatrix} 1 & 0.0005 & 0 & 0 \\ 0.0112 & 1 & -0.0007 & 0 \\ 0 & 0 & 1 & 0.0005 \\ -0.0117 & 0 & -0.0056 & 1 \end{bmatrix}, \tag{20}$$

$$B_d = \begin{bmatrix} 0 & 0 \\ 0.0049 & -0.0025 \\ 0 & 0 \\ -0.0025 & 0.0213 \end{bmatrix}, \tag{21}$$

To ensure the stability of the closed-loop system, the gain matrix K is computed using the discrete-time pole placement method, such that the eigenvalues of the matrix $A_d + B_d K$ lie within the unit circle. This placement guarantees that the closed-loop system is asymptotically stable. For this work, the desired eigenvalues were selected as $\lambda = \{0.90.950.950.9\}$, resulting:

$$K = \begin{bmatrix} 2180.5 & 32.7 & 255 & 3.8 \\ 255.5 & 3.8 & 499.7 & 7.5 \end{bmatrix}. \tag{22}$$

For the implementation of the HGO, the following parameters are considered in (7): $\varepsilon = 0.002$, $\alpha_0 = 1$, $\alpha_1 = 2$, and $\alpha_2 = 1$. Then, equations (13) and (14) can be written as:

$$S_d = \begin{bmatrix} 0.8750 & 0.0005 & 0 & 0 \\ -7.8125 & 1 & 0 & 0 \\ 0 & 0 & 0.8750 & 0.0005 \\ 0 & 0 & -7.8125 & 1 \end{bmatrix} \tag{23}$$

$$S_{Hd} = \begin{bmatrix} 0.1250 & 0 \\ 7.8125 & 0 \\ 0 & 0.1250 \\ 0 & 7.8125 \end{bmatrix} \tag{24}$$

Accordingly, by solving the discrete-time modified Francis equations given by (15), the resulting matrices Π and Γ are:

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

$$\Gamma = \begin{bmatrix} -2.1439 & 0 & 0.2993 & 0 \\ 0.2993 & 0 & 0.2993 & 0 \end{bmatrix}. \tag{26}$$

For reference tracking in the robot's workspace, and in accordance with the control objectives of this article, the required coordinate transformation from Cartesian space to joint space is performed by solving the inverse kinematics using the geometric approach [31]:

$$\theta_{1|k} = \tan^{-1}\left(\frac{p_{y|ref}}{p_{x|ref}}\right) + \tan^{-1}\left(\frac{\sqrt{1-a^2}}{a}\right), \tag{27}$$

$$\theta_{2|k} = -\tan^{-1}\left(\frac{\sqrt{1-b^2}}{b}\right), \tag{28}$$

where:

$$a = \frac{p_{x|ref}^2 + p_{y|ref}^2 + l_1^2 - l_2^2}{2l_1\sqrt{p_{x|ref}^2 + p_{y|ref}^2}}; \quad b = \frac{p_{x|ref}^2 + p_{y|ref}^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

In summary, as shown in Fig. 2, the proposed control scheme is based on the solution of the inverse kinematics of the robot. Thus, the set of solutions $[\theta_{1|k} \quad \theta_{2|k}]^T$ comprises the unmodeled discrete reference signals $[\psi_{1|k} \quad \psi_{2|k}]^T$, which are introduced into the HGO to generate the exosystem state variables $[w_{1|k} \quad w_{2|k} \quad w_{3|k} \quad w_{4|k}]^T$. Subsequently, with this information and the feedback of the plant state variables x_k, the control action u_k for each joint can be determined, in order to satisfy the regulation condition $y(k) \rightarrow y_{ref}(k)$ as $t_k \rightarrow \infty$.

V. Simulations and Results

To evaluate the performance of the proposed control scheme (Fig. 2), a plant model representation of 2-DOF robotic manipulator was designed using in Simulink® and Simscape-Multibody®, as shown in Fig. 3. The multibody model consists of five groups. Group A contains the simulation settings, including the solver configuration, the definition of the gravity vector, and the World reference frame. Groups B and C correspond to the first and second kinematic pairs: the first revolute joint connects the World frame to Link 1, while the second joint connects Link 1 to Link 2. Joints are configured to be actuated by the control torques and provide both angular position and velocity feedback. The physical properties of each link, such as geometry, mass, and moments of inertia, are defined using Solid blocks. Finally, Group D includes the visualization components: a spline curve that represents the desired Cartesian trajectory based on reference points, and a dummy solid associated to the end of Link 2. For further details about Simscape multibody modeling, refer to [32].

A. *Case I: Tracking of a Chaotic Dynamical System*

As first case of study is considered a reference signal in Cartesian coordinates given by the Lorenz attractor (Fig. 1). Lorenz attractor is characterized by its chaotic dynamics, sensitivity to initial conditions and unpredictable behavior, making it ideal to demonstrate the robustness of proposed control approach.

$$\begin{aligned} w_{k+1} &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + T_s \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -w_{1,k} \\ w_{2,k} & 0 & -\frac{8}{3} \end{bmatrix} \right) \begin{bmatrix} w_{1,k} \\ w_{2,k} \\ w_{3,k} \end{bmatrix} \\ y_{ref} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_{1,k} \\ w_{2,k} \\ w_{3.k} \end{bmatrix} \end{aligned} \tag{29}$$

The output of the Lorenz attractor y_{ref} provides the Cartesian reference trajectory used to solve the inverse kinematics of the manipulator, such that $y_{ref} = [P_{x|ref} \quad P_{y|ref}]^T$. Despite $[P_{x|ref} \quad P_{y|ref}]^T$ are given by exosystem (29), the joint coordinates $\theta_{i|k}$ lack of an explicit analytical model. Figs. 4a and 4b depict the estimation and tracking of the reference signal for joint 1 and joint 2, respectively. While, in Figs. 4c and 4d are appreciated the tracking errors in robot joint space, with a maximum error of 0.13 rad during a transitory of 0.5 s. Accordingly, the above joint errors produce the reference tracking in the workspace as shown in Fig. 5a with a bounded error of 0.04 m (Fig. 5b). The control signals u_1 and u_2 for the tracking are shown in Fig. 6.

B. *Case II: Tracking of a Prescribed Workspace Trajectory*

Tasks such as assembly, manipulation, and welding, are implemented as prescribed trajectories within the robot's workspace to facilitate the programming process. Rectangular trajectories are commonly used for these tasks. As shown in Fig. 7, a trajectory composed of five linear segments is defined in the Cartesian workspace. To specify the desired position

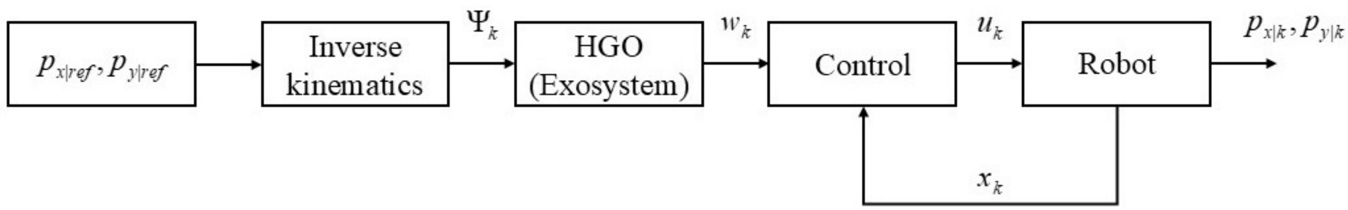


Fig. 2. Block diagram of the control scheme for tracking unmodeled reference signals in a robotic manipulator.

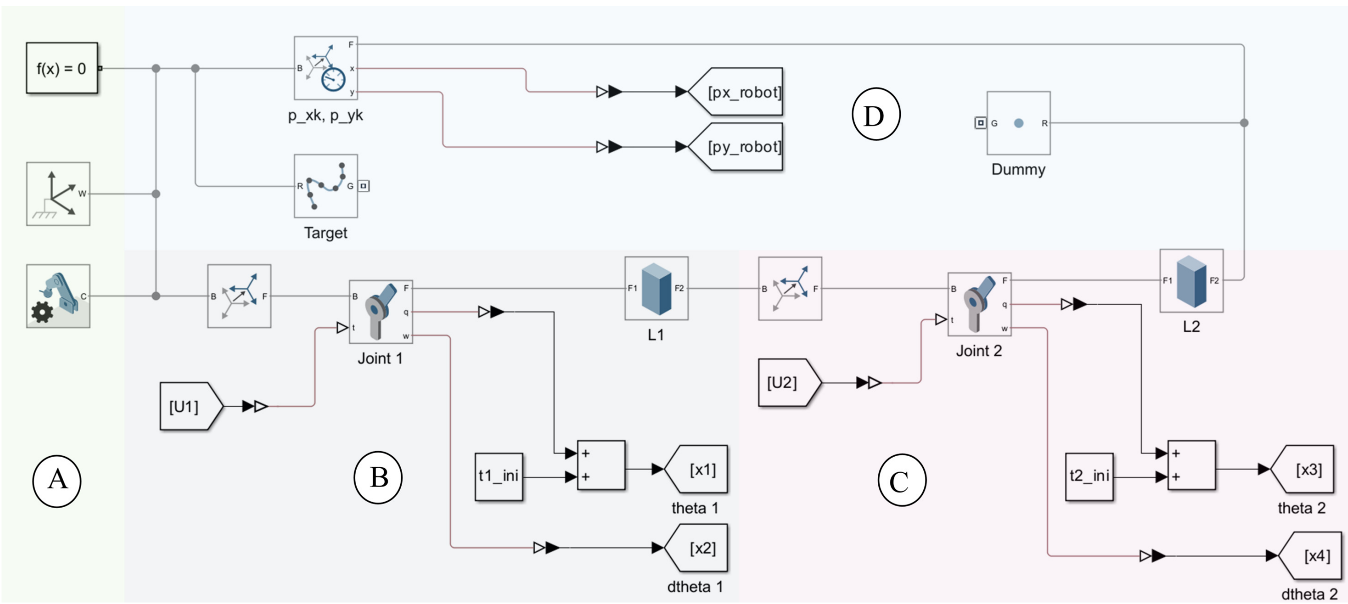


Fig. 3. Block diagram of a 2-dof robotic manipulator in Simscape-Multibody

${}^{0}\vec{p}_{2,j}$ and orientation ${}^{0}R_{2,j}$ of the gripper with respect to the base frame, homogeneous transformation matrices ${}^{0}T_{2,j}$ are defined for each via point j, where $j = 0, \ldots, 4$. Intermediate positions and orientations between two consecutive points ${}^{0}\vec{p}_{2,j}, {}^{0}R_{2,j}$ and ${}^{0}\vec{p}_{2,j+1}, {}^{0}R_{2,j+1}$ are calculated using linear and spherical interpolation [33], respectively.

$$ {}^{0}T_{2,1} = \begin{bmatrix} 0.9133 & 0.4073 & 0 & 0.4949 \\ -0.4073 & 0.9133 & 0 & 0.2949 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{0}T_{2,2} = \begin{bmatrix} 0.9742 & -0.2258 & 0 & 0.60 \\ 0.2258 & 0.9742 & 0 & 0.60 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{0}T_{2,3} = \begin{bmatrix} 0.9239 & -0.3827 & 0 & 0.75 \\ 0.3827 & 0.9239 & 0 & 0.60 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{0}T_{2,4} = \begin{bmatrix} 0.5876 & 0.8091 & 0 & 0.75 \\ 0.8091 & 0.5876 & 0 & -0.20 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

$$ {}^{0}T_{2,5} = \begin{bmatrix} 0.3552 & 0.9348 & 0 & 0.60 \\ -0.9348 & 0.3552 & 0 & -0.20 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

In Figs. 8a and 8b is shown how the exosystem state variables w_1 and w_3 accurately estimate the reference signals Ψ_1 and Ψ_2 for the rectangular trajectory. Subsequently, the plant states x_1 and x_3 track w_1 and w_3, maintaining bounded errors (Figs. 8c and 8d). In accordance with the Francis' theory, the regulation condition in the workspace is satisfied as can be observed in Fig. 9a and 9b, the control signals to generate the desired motion are presented in Fig. 10.

C. Case III: Tracking of a Reference Signal Generated in Real Time

A potential application of the proposed controller is the tracking of reference signals generated in real time during human-robot interaction, where free-form trajectories can draw by an user. In addition to the absence of a reference signal model, the presence of abrupt curvature changes and high-frequency components poses a challenge for the regulation task. In this case, the reference trajectory was generated using an input device (pen tablet) with a sampling time of $T_s = 5 \times 10^{-5}$ s. The acquired signal is shown in Fig. 11

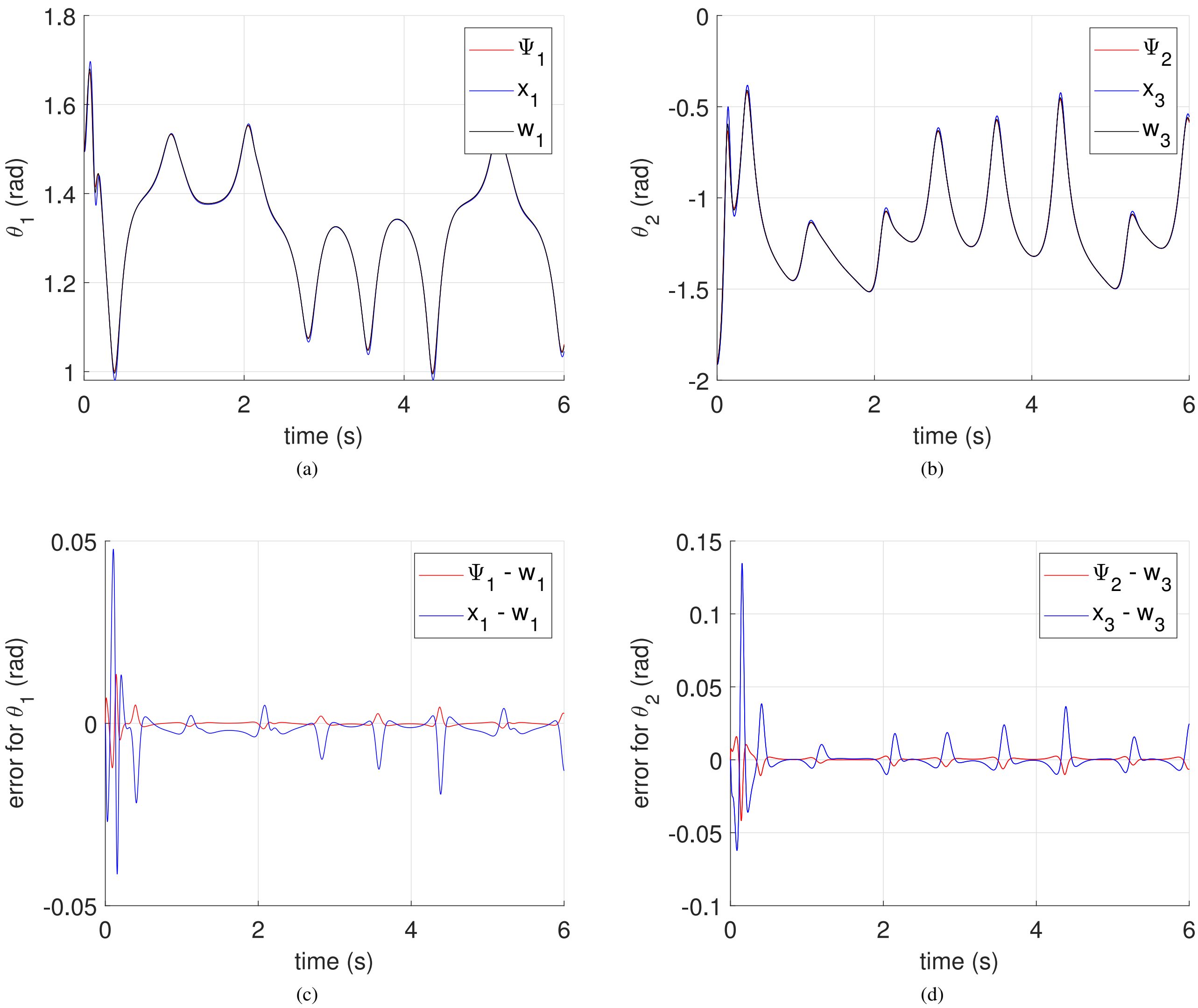
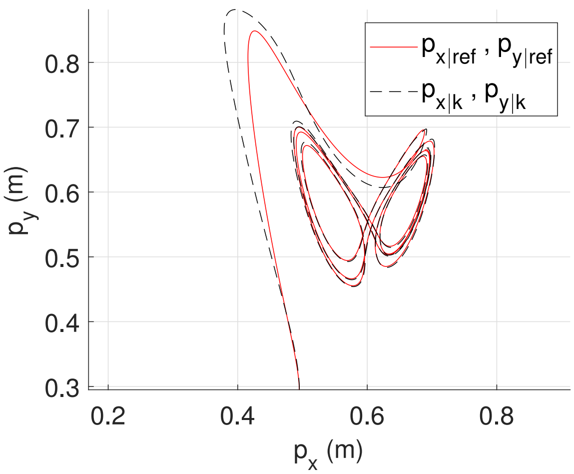


Fig. 4. Estimation and tracking results in the robot joint space. a) Reference signal Ψ_1 vs. robot state x_1 vs. exosystem state w_1. b) Reference signal Ψ_2 vs. robot state x_3 vs. exosystem state w_3. c) Estimation error of the reference signal ($\Psi_1 - w_1$) and tracking error in joint 1 ($x_1 - w_1$). d) Estimation error of the reference signal ($\Psi_2 - w_3$) and tracking error in joint 2 ($x_3 - w_3$).

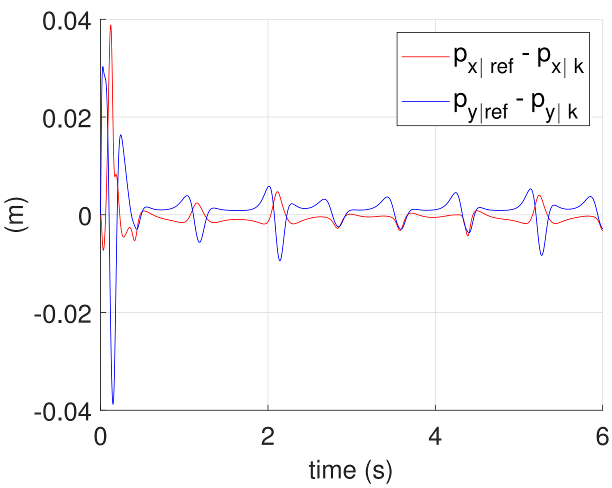
The results for state estimation, trajectory tracking, and joint-space errors are presented in Figs.12a - 12d. Fig.13 illustrates the tracking performance in the robot's workspace, where the error remains bounded, similar to the previous cases. Finally, Fig. 14 displays the control inputs applied to joint 1 and joint 2.

VI. Conclusions

This work presented the tracking of unmodeled signals using Francis' linear output regulation theory in robotic manipulators. It is assumed that the reference signal lies in the robot's workspace, and by performing the corresponding coordinate transformation, the reference in the joint space does not have a defined model. Since the trajectory is obtained from a series of discrete points, the necessary conditions to solve the output regulation problem are established through the modified discrete Francis equations, due to the integration of the high-gain observer. Despite that the controller is designed under the assumption of the linear model, it was observed that the tracking error in each joint remains bounded, which is also reflected in the workspace. It is important to note that, due to the use of high-gain observers (HGOs), filtering the signals to be estimated is strongly recommended; otherwise, the stability of the closed-loop system may be compromised. Moreover, both the robustness to noise and the convergence rate of the HGO are influenced by the parameter ε. Consequently, a careful trade-off must be made in each application to balance fast convergence against sensitivity to noise. Finally, the development of this work demonstrated that the high-gain observer, used as an exosystem, represents an alternative for tracking discrete unmodeled signals, opening the possibility of extending this technique to various areas of robotics such as computer vision, haptic interfaces, mobile robotics, and different real-time applications.



(a)



(b)

Fig. 5. Results for tracking in the robot workspace for Case I. a) Tracking of trajectory in the workspace. b) Tracking error in the workspace.

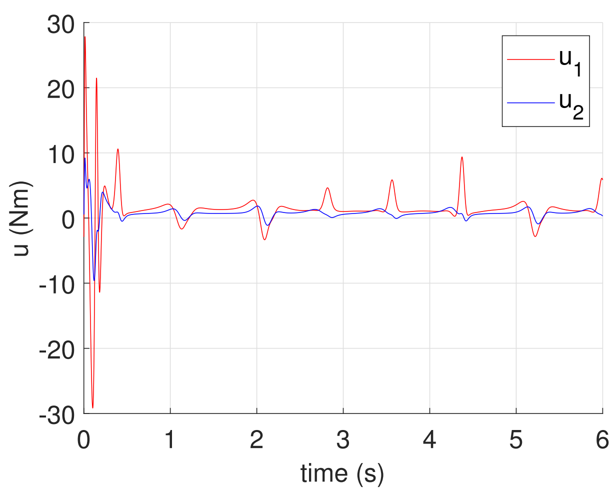


Fig. 6. Control signals for the tracking of Lorenz attractor

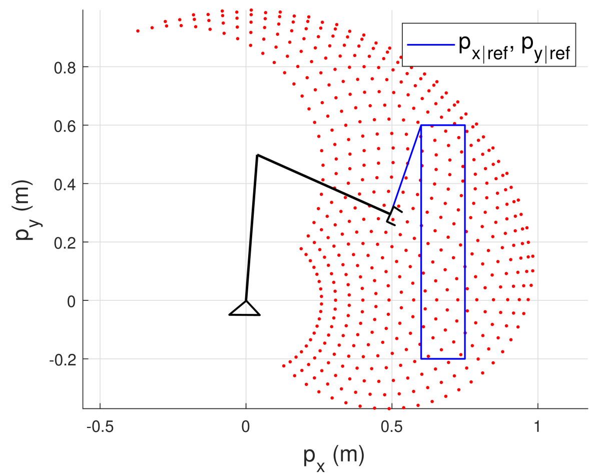


Fig. 7. Case II: Reference trajectory.

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Appendix A
Comparison of Control Approaches for Output Regulation with Unknown Reference Signals

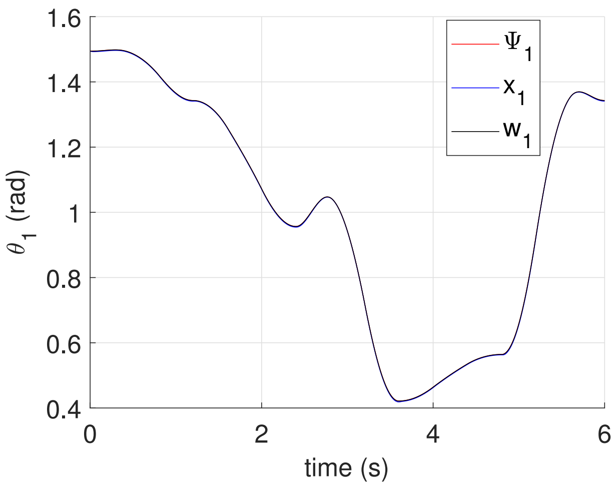
This section briefly describes several approaches used in scenarios where the reference signals are unknown a priori. Table I summarizes the key advantages and disadvantages of each control strategy when applied to output regulation under such conditions.

A. Output Regulation with High-Gain Observer-based Exosystem Estimation

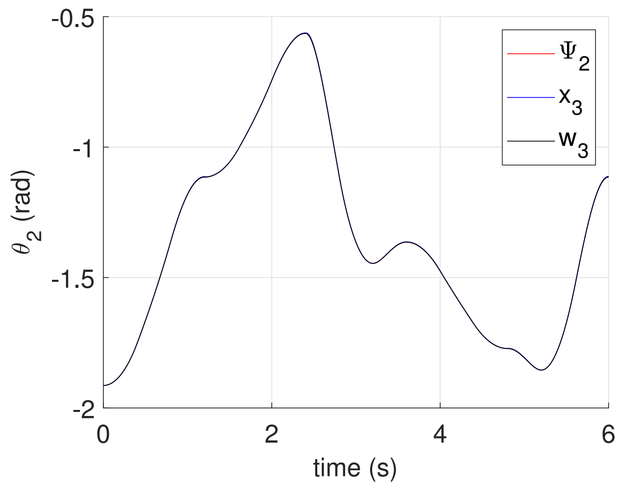
This approach directly addresses the problem of unknown a priori reference or disturbance signals by employing high-gain observers (HGOs) to estimate these unmodeled but measurable signals. The estimated signals are then used to construct an effective exosystem, which allows for the application of modified Francis equations to achieve output regulation. This extends classical regulation results to scenarios where the exogenous signals' dynamics are not fully known [21], [23].

B. Adaptive Internal Model Principle

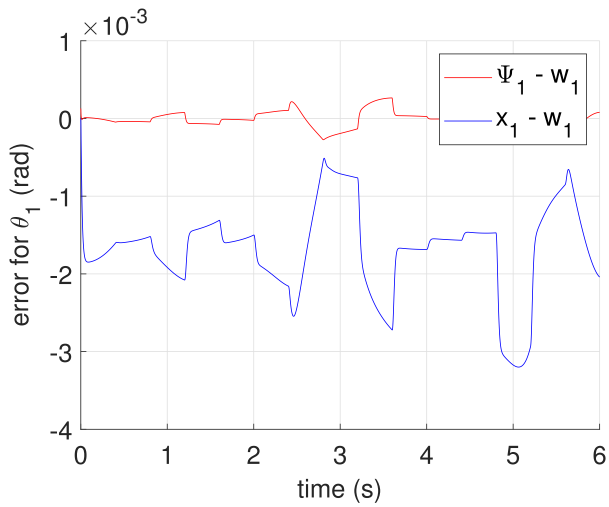
The Adaptive Internal Model Principle (AIM) extends the classical Internal Model Principle by incorporating adaptive observers to estimate unknown parameters of the exosystem,



(a)



(b)



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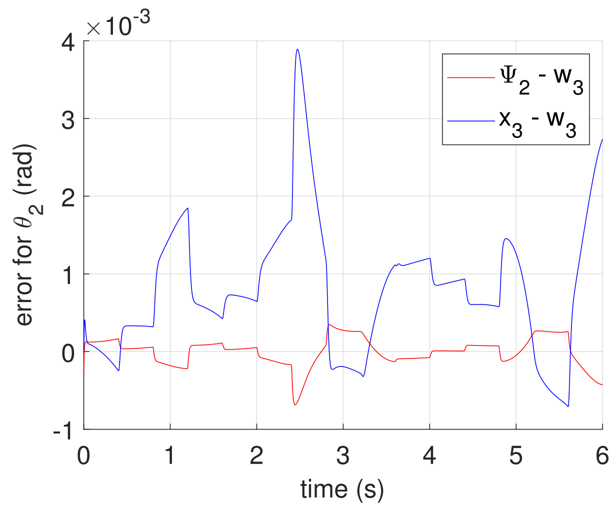


(d)

Fig. 8. Estimation and tracking results in the robot joint space. (a) Reference signal Ψ_1 vs. robot state x_1 vs. exosystem state w_1. (b) Reference signal Ψ_2 vs. robot state x_3 vs. exosystem state w_3. (c) Estimation error of the reference signal ($\Psi_1 - w_1$) and tracking error in joint 1 ($x_1 - w_1$). (d) Estimation error of the reference signal ($\Psi_2 - w_3$) and tracking error in joint 2 ($x_3 - w_3$).

such as frequencies and amplitudes of sinusoidal signals. Once these parameters are estimated, an internal model can be dynamically constructed or adapted within the controller to achieve the desired regulation. This approach is particularly effective for unknown harmonic reference signals and can transform the output regulation problem into a stabilization problem [7], [8], [34], [35].

C. Data-Driven Control and Learning-Based Methods

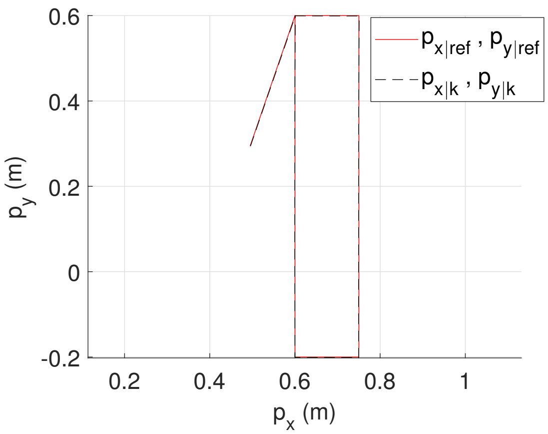
These modern paradigms address control problems without requiring explicit knowledge of the system's dynamics or precise models of exogenous signals. Instead, they learn control policies directly from collected data [36], [37].

D. Non-Adaptive Robust Control with Generic Internal Models

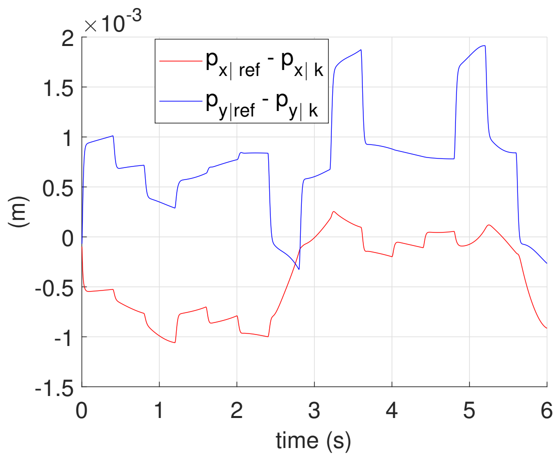
Instead of explicit adaptation, this methodology designs controllers that are inherently robust to unknown exosystem dynamics. It constructs "generic internal models" for the system's steady-state variables, converting the robust output regulation problem into a non-adaptive stabilization problem for an augmented system. This approach aims to guarantee robustness with respect to unmodeled disturbances and unknown exosystem dynamics without explicitly identifying them, thereby avoiding phenomena like "bursting" often associated with adaptive control [38].

E. Active Disturbance Rejection Control (ADRC)

Active Disturbance Rejection Control (ADRC) is a model-free control technique that employs an Extended State Observer (ESO) to estimate and compensate for "total distur-



(a)



(b)

Fig. 9. Results for tracking in the robot workspace for Case II. (a) Tracking of trajectory in the workspace. (b) Tracking error in the workspace.

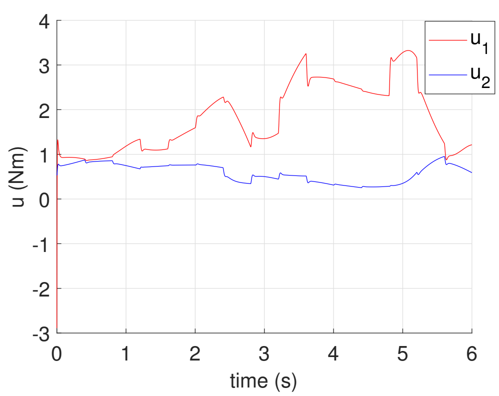


Fig. 10. Control signals for the tracking of Case II

bance," which includes external disturbances, unknown internal dynamics, and the unknown reference signal itself. This allows the controller to operate effectively even with a very simple nominal model of the plant, as the unknown reference signal is treated as part of this total disturbance [39].

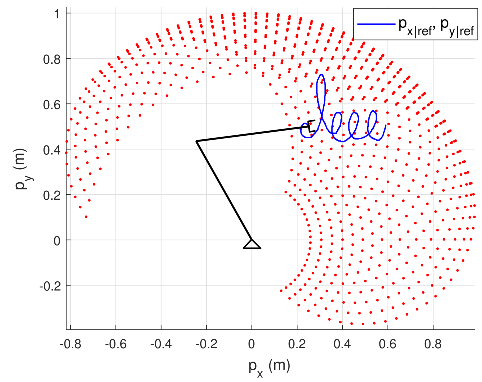


Fig. 11. Case III: Reference trajectory.

TABLE I
SUMMARY OF CONTROL APPROACHES FOR OUTPUT REGULATION WITH UNKNOWN REFERENCE SIGNALS

Approach	Advantages	Disadvantages
Output Regulation with HGO-based Exosystem Estimation	Extends classical output regulation to unknown reference/disturbance signals. Systematically applies Francis equations via estimated exosystem. Enables regulation for nonlinear systems with minimal bounded error. Simple and implementable controller design.	Peaking phenomenon: large transient errors/inputs. Sensitive to sensor noise due to high gains. Potential computational load for high-dimensional systems.
Adaptive Internal Model Principle	Manages unknown reference frequencies/amplitudes. Converts regulation into stabilization. Can achieve exponential convergence. Does not require full state measurements.	Some methods assume known number/frequency structure. May require system-specific transformations.
Data-Driven Control (ADP/RL, Data-Driven IMC)	No prior model knowledge required. Handles noisy data. Can achieve optimal transient and steady-state performance. Allows exact regulation in LTI systems. Adapts to unknown or time-varying parameters.	Observer convergence may be required for performance guarantees. Stability proofs can be complex. Computationally intensive.
Non-Adaptive Robust Control with Generic Internal Models	Robust to unmodeled dynamics and unknown disturbances. Can ensure global asymptotic stability.	Can be conservative (worst-case design). May require conditions for internal model existence.
Active Disturbance Rejection Control (ADRC)	Model-free structure. Real-time estimation of total disturbance. Robust to model uncertainty. Simple tuning and implementation. High disturbance attenuation.	Estimation error is bounded, not zero, for non-constant disturbances. Basic ESO may not fully capture oscillatory or structured disturbances.

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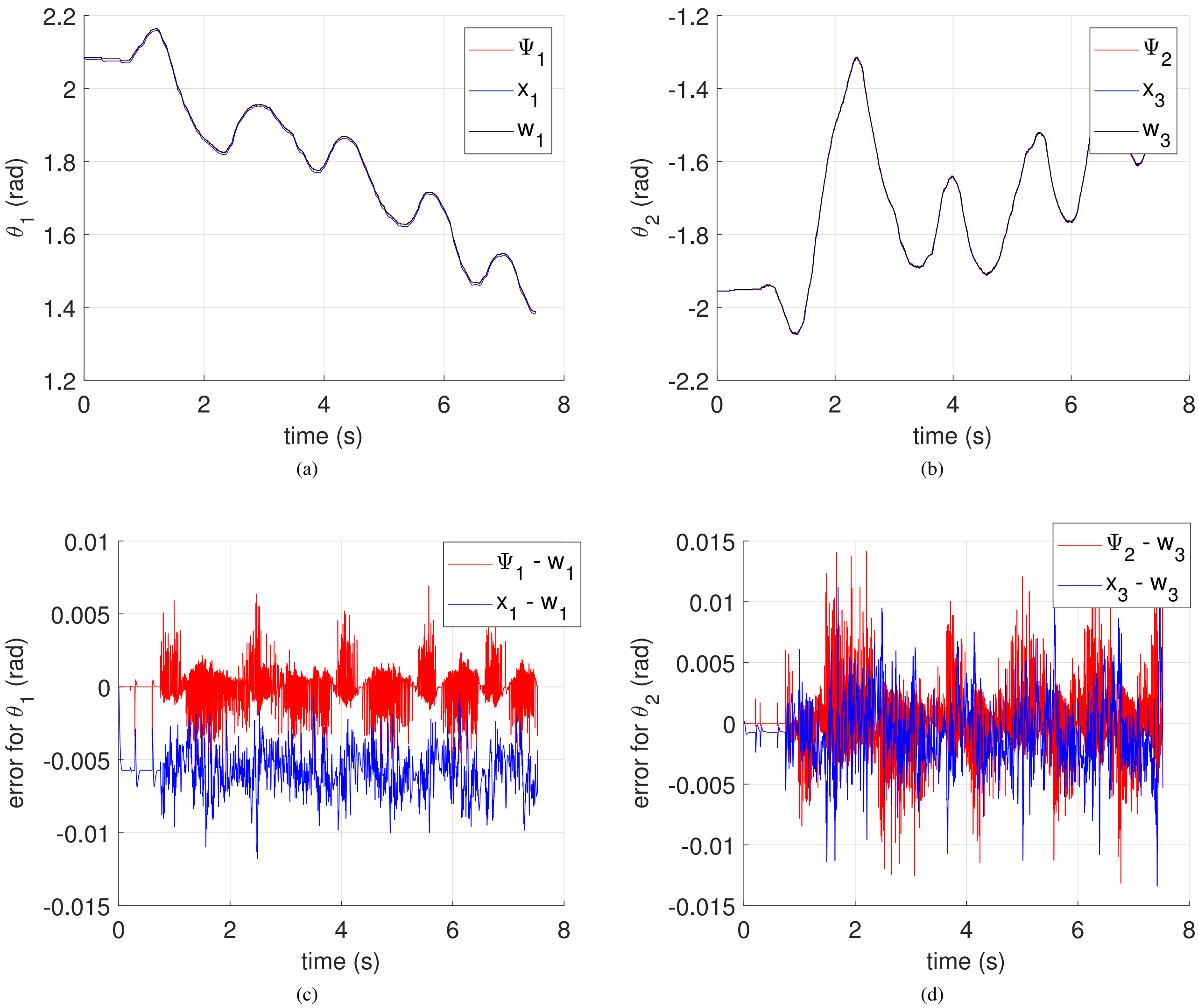

Fig. 12. Estimation and tracking results in the robot joint space. (a) Reference signal Ψ_1 vs. robot state x_1 vs. exosystem state w_1. (b) Reference signal Ψ_2 vs. robot state x_3 vs. exosystem state w_3. (c) Estimation error of the reference signal $(\Psi_1 - w_1)$ and tracking error in joint 1 $(x_1 - w_1)$. (d) Estimation error of the reference signal $(\Psi_2 - w_3)$ and tracking error in joint 2 $(x_3 - w_3)$.

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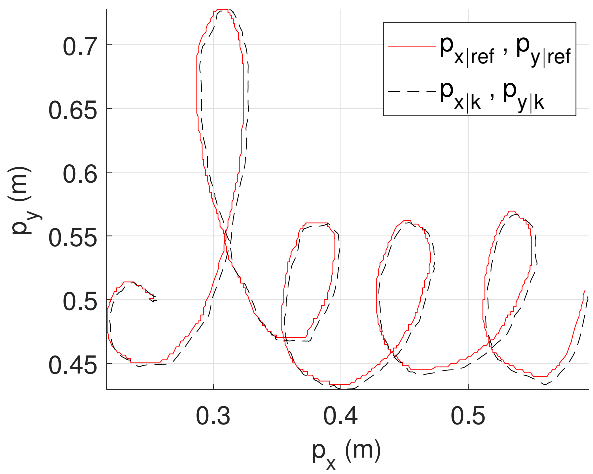
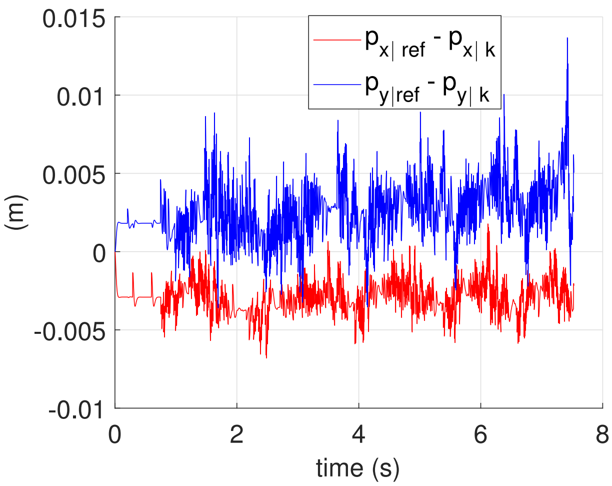


(a)



(b)

Fig. 13. Results for tracking in the robot workspace for Case III. (a) Tracking in the workspace. (b) Tracking error in the robot workspace.

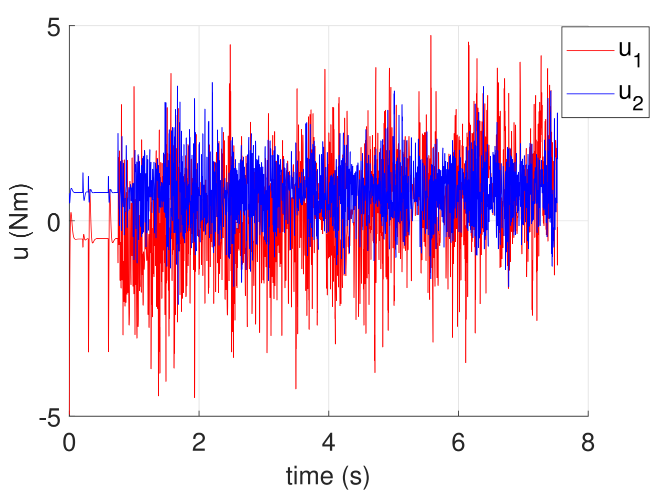


Fig. 14. Control signals for the tracking of Case III

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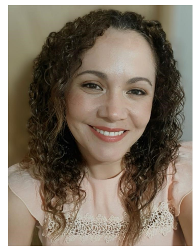

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