Novel Range Wise Optimization of the Exponential Bounds on the Gaussian Q Function and its Applications in Communications Theory

Aditya Powari, Garv Anand and Dharmendra Sadhwani

Abstract—**This paper presents a novel and highly effective method for improving the accuracy of approximations for the Gaussian Q function. By rigorously optimizing the coefficients of the approximations using the interior point optimization technique, significantly tighter bounds are achieved with simplicity intact. The proposed approach, which is applicable to a wide range of scenarios, focuses on enhancing the simple exponential bounds proposed in the literature. Through a comprehensive analysis based on the relative error, the superiority of the optimized coefficients compared to the existing bounds and approximations available in the open literature is demonstrated. Moreover, an insight into the generic applicability of the optimized coefficients is provided, which exhibits excellent performance in terms of the absolute error as well. The Gaussian Q function plays a crucial role in evaluating the performance of diverse wireless communication systems under various challenging fading distributions. Therefore, the proposed research significantly contributes to advancing the accuracy of the approximations of the Gaussian Q function, enabling improved error performance for coherent digital modulation techniques. The findings presented herein offer valuable contributions to the state-of-the-art and set a new standard for accuracy in the work related to Gaussian Q function approximations.**

Index Terms—**Optimization algorithm, Gaussian Q function, Wireless communication systems, Performance evaluation, approximate computing, Digital modulation techniques, Fading channels, Bit-error rate, Symbol-error rate**

I. Introduction

In exact form, the Gaussian Q function [1] plays a vital role in performance analysis of wireless communication systems over additive white Gaussian noise and fading channels [1]. However, this form is not tractable when it comes to compute the key metrics like symbol error probability (SEP) of several coherent digital modulation schemes like filtered multitone modulation [2] over complex fading distributions like: Fluctuating Beckmann fading [3], $\kappa-\mu$ shadowed fading channel [4], cascaded double $\kappa-\mu$ shadowed fading channel [5] and Log-normal distribution [6], resulting into cumbersome double definite integrals. Hence, it is necessary to simplify the exact form of the Gaussian Q function into some tractable forms. To meet this requirement, several approximations/bounds of the Gaussian Q function are available in the literature [7]–[24].

The analysis corresponding to the accuracy and the tractability of all the aforementioned approximations is a topic of interest. For example, the authors in [8] derived an approximation which is tight for the lower values of x; whereas its accuracy deteriorates as the value of x increases. Moreover, the approximation [8] is also not tractable enough due to the presence of intractable forms of the algebraic and exponential functions. The intractability of [8] is somewhat handled in [9] where one of the exponential terms of [8] is expanded using Taylor's series method yielding an approximation whose accuracy is dependent on the number of terms i.e. only a certain level of accuracy is achieved in [9] that too with a large number of terms (higher computational complexity). Hence, it is evident that if one tries to improve the accuracy, the price to be paid is increased computational complexity and the vice-versa. Apart from this, in the literature it has been observed that the approximations often yield accurate results for one particular range of x, due to which the accuracy at other values of x is compromised; which can be seen in [19]. Noteworthy, the authors in [19] optimized an already existing approximation of the Gaussian Q function on the basis of absolute error (AE) and the relative error (RE) but it has been observed that while optimizing the RE, the AE increases which further limits the significance of [19] as the optimized coefficients should be versatile in nature i.e. they must yield desired results for both the AE and the RE. Further, using the trapezoidal rule of integration, the authors in [17] have proposed accurate yet simple exponential based approximations of the Gaussian Q function. Since the approximations are expressed as the sum of simple exponentials, they prove to be significant while solving the cumbersome SEP integrals of various digital modulation schemes over intractable fading statistics. It should be noted that although the trapezoidal rule of numerical integration is considered as a fairly accurate method, the approximations obtained in [17] do not yield desired results particularly for very low values of x i.e. $x \leq 0.5$ and for a higher range $x \geq 3$.

In view of the above stated challenges, optimization methods based on Interior point algorithm prove to be significant. These are a class of optimization algorithms which are used to solve constrained optimization problems [25]. These methods have been widely used in various fields like sparse signal reconstruction [26], network constrained security control problem [27], optimal power flow problems [28], quantile regression [29], localization problem of wireless sensor network [30]

Aditya Powari is with the Department of Electrical and Electronic Engineering, The University of Manchester, M13 9PL Manchester, U.K. (e-mail: adityapowari@gmail.com). Garv Anand and Dharmendra Sadhwani are with the department of Electrical and Computer Science Engineering, IITRAM, Ahmedabad, 380026, India. (e-mail: garv.anand.20e@iitram.ac.in, dharmendrasadhwani@gmail.com.

[1]
$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\tfrac{u^2}{2})du, \tag{1}$$

and optimization of the exponential bounds on the Gaussian Q function [31].

Motivated by this, in this paper, a generic methodology is developed which performs rigorous range wise optimization on the already existing simple exponential bounds on the Gaussian Q function. For demonstration purpose, in this paper, the bounds proposed by the authors in [17] are optimized. Precisely, the RE for various ranges of x is optimized to obtain more accurate optimized approximations in terms of the new optimized coefficients. To do so, Non-Linear Multivariate Optimization technique based on the interior-point algorithm is implemented.

II. Mathematical Background on Optimization using Interior Point Algorithm

In interior point optimization, the objective is to find the optimal solution within the feasible region of a problem while satisfying the constraints.The key idea behind interior point methods is to transform the constrained optimization problem into an unconstrained problem by introducing a barrier or penalty function that penalizes solutions outside the feasible region. The generalized mathematical background for the interior point algorithm is elaborated below:

- Defining the objective function for the problem, and listing the constraints involved [2](both equality and inequality constraints), as shown below:

$$\begin{aligned} \min_{x\in\mathbb{R}} \quad & RE(x) \\ \text{such that} \quad & g(x) \geq b, \\ & h(x) = 0, \end{aligned} \tag{2}$$

  where $RE(x)$ is the relative error which serves as an objective function, $g(x)$ is the inequality constraint, and $h(x)$ is the equality constraint.
  The motive is to minimize the $RE(x)$, which is defined as:

$$RE(x) = \frac{|erfc(x) - F(x)|}{erfc(x)}, \tag{3}$$

  where $F(x)$ denotes new optimized approximated function, $erfc(x)$ is the complementary error function evaluated for each element of x.

- After defining the general non-linear problem as described in (2), the formulation is converted into a general form using slack variables. The final set of objective functions and constraints can be represented as follows:

$$\begin{aligned} \min_{x\in\mathbb{R}} \quad & RE(x) \\ \text{such that} \quad & c(x) = 0, \\ & \text{for } x \geq 0. \end{aligned} \tag{4}$$

- Once the general form (4) is obtained, a barrier function needs to be defined to eliminate the inequality constraints. This barrier function introduces a penalty term that penalizes solutions outside the feasible region, effectively transforming the constrained optimization problem into an unconstrained one. This can be illustrated using the equations provided below:

$$\begin{aligned} \min_{x\in\mathbb{R}} \quad & RE(x) - \mu \sum_{i=1}^{n} \ln(x_i), \\ \text{such that} \quad & c(x) = 0. \end{aligned} \tag{5}$$

  The typical choice for the Barrier is a log barrier. Natural log barrier terms have been specified for inequality constraints, where μ represents the weight associated with the barrier. This replaces the hard inequality constraints with a smoother objective function. As $x_i \to 0$, $\ln(x_i) \to\infty$, so a very small value of μ will lead to convergence very close to the constraints. From Fig. 1, it can be clearly seen that as the value of μ decreases, a smoother curve is obtained.

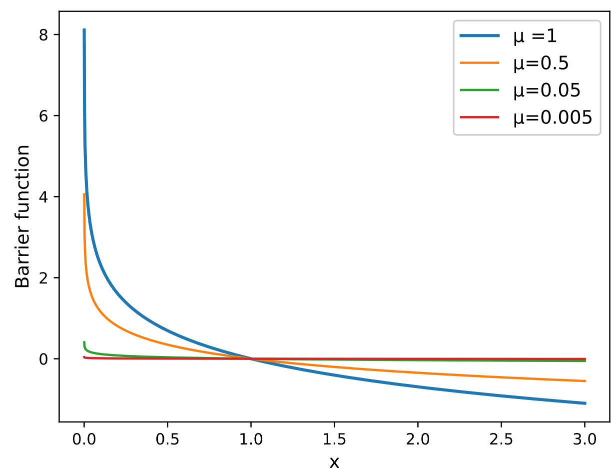


Fig. 1. The barrier function vs x.

- In order to solve the barrier functions, the Karush–Kuhn–Tucker conditions (KKT), which are first derivative tests (sometimes referred to as first-order necessary conditions) for a solution in nonlinear programming to be deemed optimal, provided that certain regularity conditions are satisfied. The KKT conditions for the barrier function (5) are as follows:

$$\left\{ \begin{aligned} \min_{x\in\mathbb{R}} \quad & RE(x) - \mu \sum_{i=1}^{n} \ln(x_i) \\ \text{such that} \quad & c(x) = 0 \end{aligned} \right. \implies \left\{ \begin{aligned} & \nabla RE(x) + \\ & \nabla c(x)\lambda - \\ & \mu \sum_{i=1}^{n} \frac{1}{x_i} = 0 \end{aligned} \right.$$

  where λ denotes the Lagrangian multiplier. Now replacing $z_i = \frac{\mu}{x_i}$ and solving the modified version of the KKT conditions, we have

$$\nabla RE(x) + \nabla c(x)\lambda - z = 0, \tag{6}$$

$$c(x) = 0, \tag{7}$$

$$(X \cdot Z \cdot e) - (\mu \cdot e) = 0. \tag{8}$$

[2]The proposed method does not have any set of constraints

Eq. (8) is obtained by the replacement of $z_i = \frac{\mu}{x_i}$,where, e is column matrix of all ones i.e.

$$\left.\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\right\} n$$

- Solving the barrier function, i.e finding KKT solution with Newton Raphson method (to find the search direction will be the next step):

$$\begin{bmatrix} W_k & \nabla c(x_k) & -I \\ \nabla c(x_k)^T & 0 & 0 \\ Z_k & 0 & X_k \end{bmatrix} \cdot \begin{bmatrix} d_k^x \\ d_k^\lambda \\ d_k^Z \end{bmatrix} = -\begin{bmatrix} \nabla RE(x_k + \nabla c(x_k)\lambda_k - z_k \\ c(x_k) \\ X_k \cdot Z_k.e - \mu_j \cdot e \end{bmatrix}, \tag{9}$$

where W_k is the second derivative of the Lagrangian and is denoted by $W_k = \nabla^2_{xx} L(x_k, \lambda_k, z_k) = \nabla^2_{xx}(RE(x_k) + c(x_k)^T \lambda_k - z_k)$,

$$Z_k = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_n \end{bmatrix}$$

and

$$X_k = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_n \end{bmatrix}.$$

- The process involves re-arranging the system into a symmetrical linear form ($A \cdot X = B$). This matrix can then be solved using a linear symmetric solver capable of handling matrices in the form of $A \cdot X = B$, as

$$\begin{bmatrix} W_k + \sum_k & \nabla c(x_k) \\ \nabla c(x_k)^T & 0 \end{bmatrix} \cdot \begin{bmatrix} d_k^x \\ d_k^\lambda \end{bmatrix} = -\begin{bmatrix} \nabla RE(x_k + \nabla c(x_k)\lambda_k \\ c(x_k) \end{bmatrix},$$

where $\sum_k = X_k^{-1} Z_k$.
- Solve for d_k^Z after the linear solution to d_k^x and d_k^λ with explicit solution:

$$d_k^z = \mu X_k^{-1} e - z_k - \sum_k d_k^x.$$

- Step Size: The determination of step size is a critical step in the optimization process. After obtaining the search direction, it's essential to identify the appropriate step size. There are two main approaches to assessing progress. The first approach involves minimizing the objective function, while the second focuses on minimizing constraint violation. Additionally, two popular strategies can be employed: decreasing the merit function and using the filter method. Once the approaches have been considered, the next step is to adjust the variables x, λ, and z based on the determined step size. This involves updating them according to the following equations:

$$x_{k+1} = x_k + \alpha_k d_k^x, \tag{10}$$
$$\lambda_{k+1} = \lambda_k + \alpha_k d_k^\lambda, \tag{11}$$
$$z_{k+1} = z_k + \alpha_k d_k^z. \tag{12}$$

Here, $(x_{k+1}, \lambda_{k+1}, z_{k+1})$ and (x_k, λ_k, z_k) represent the values of x, λ, and z at the $k+1$ and k iterations respectively. The parameter α_k denotes the step size.

- Convergence Criteria: Convergence is achieved when the KKT conditions are satisfied within the specified tolerance, and the desired values of minima are attained.

$$\max\{|\nabla RE(x) + \nabla c(x)\lambda - z|\} \leq \epsilon_{tot}, \tag{13}$$
$$\max\{c(x)\} \leq \epsilon_{tot}, \tag{14}$$
$$\max\{|(X \cdot Z \cdot e) - (\mu \cdot e)|\} \leq \epsilon_{tot}, \tag{15}$$

where ϵ_{tot} denotes the acceptable tolerance (defined as per requirement).
A detailed flow diagram of the interior points algorithm has been shown in Fig. 2.

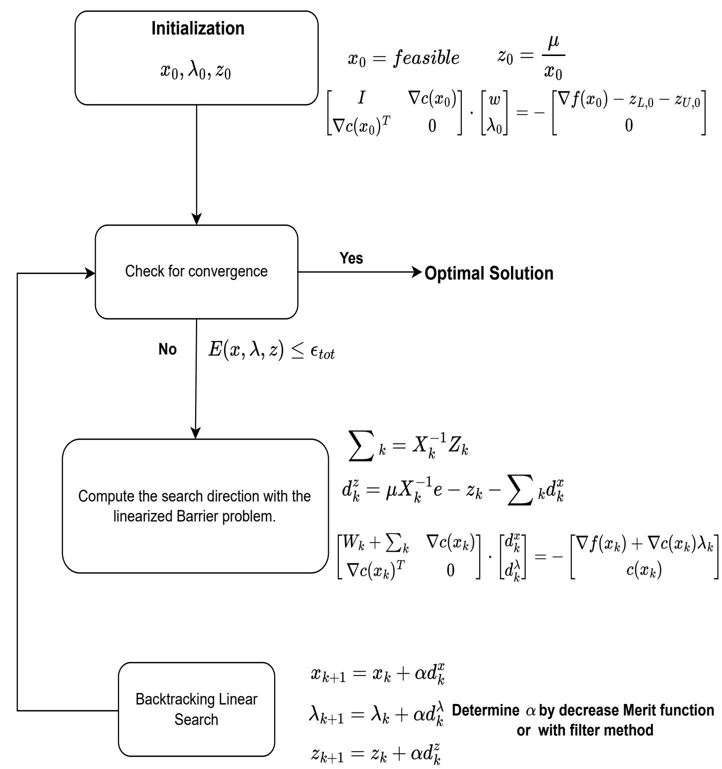


Fig. 2. Flowchart of Interior Points Algorithm

III. Novel Range Wise Optimization and New Optimized Coefficients Using Interior Point Algorithm

Using the trapezoidal rule of numerical integration, for three ($n = 3$) and four ($n = 4$) subintervals, the authors in [17] have proposed tight exponential bounds on the complementary error function respectively defined as:

$$F(x) \approx \frac{1}{6}\exp(-x^2) + \frac{1}{3}\exp(-4x^2) + \frac{1}{3}\exp(-1.33x^2) \tag{16a}$$

and

$$F(x) \approx \frac{1}{8}\exp(-x^2) + \frac{1}{4}\exp(-2x^2) + \frac{1}{4}\exp(-6.67x^2) + \frac{1}{4}\exp(-1.176x^2). \tag{16b}$$

The coefficients of the exponential terms of (16) viz. $\left[\frac{1}{6}, \frac{1}{3}, \frac{1}{3}\right]$ for $n = 3$ and $\left[\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right]$ for $n = 4$ are not providing optimum results for a considerable range of x. This is evident by the occasional 'dips' seen in the RE curve of the original approximations over a range $x \in [0, 5]$ as shown in Fig 3. The RE is calculated as:

$$RE(x) = \frac{|erfc(x) - F(x)|}{erfc(x)}, \tag{17}$$

where

$$F(x)_{(n=3)} \approx \alpha\exp(-x^2) + \beta\exp(-4x^2) + \zeta\exp(-1.33x^2) \tag{18a}$$

and

$$F(x)_{(n=4)} \approx \alpha'\exp(-x^2) + \beta'\exp(-2x^2) + \zeta'\exp(-6.67x^2) + \delta'\exp(-1.176x^2). \tag{18b}$$

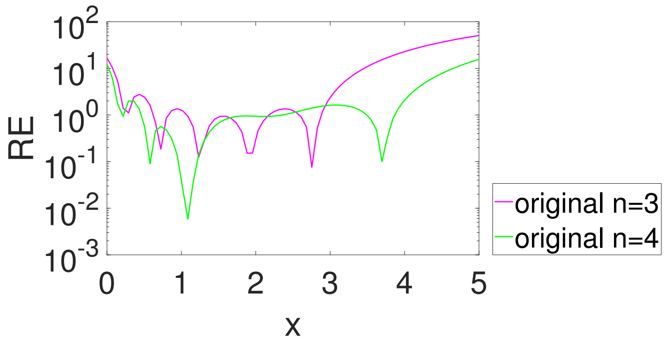


Fig. 3. RE Plot for original approximation (16).

Referring to Fig. 3, the region around which these 'dips' occur basically signifies the regions where the original coefficients yield accurate results as the RE is getting reduced in those regions. However, the RE rises considerably in the regions which lie beyond these 'dips'. The frequency of these 'dips' is very low which implies that the original set of coefficients are not performing well for quite a considerable range of x. This motivates the authors in the present work to carry out range-wise optimization of the original approximation (16) by defining a new set of coefficients for the exponential terms as α, β, ζ for $n = 3$ and α', β', ζ', δ' for $n = 4$; giving rise to new optimized approximations. The idea here is to carry out range-wise optimization of (16) i.e. for each specific range of x the best optimum coefficients were obtained for the exponential terms which further enhances the performance of (16) by reducing the RE even further. Moreover, the proposed range-wise optimization technique is not affected by the widely recognized problem which is minimization algorithm getting stuck in a $local - minima$ rather than the $global - minima$. Hence, the total range of x (which is to be optimized) is divided into several smaller ranges, effectively avoiding this problem. For each new range of x, the best optimal coefficients were obtained. This division has been done by getting an understanding of the regions in which the original coefficients are not performing well which is quite evident from Fig. 3. Furthermore, for each different value of x, the behavior of the exponential terms used in (16) will change accordingly and hence it is imperative that the coefficients should also be updated in such a way as to get the best possible optimum results for any value of x taken. In the proposed investigation, ϵ_{tot}, denoting the acceptable tolerance, is construed as a parameter subject to user specification. Specifically, the initial coefficients, as derived from the work of [17], yielded an error of 10^{-1} which is 10% for $n = 3$. However, in pursuit of heightened precision and an error rate of less than 1%, the proposed work judiciously adjusted the tolerance to approximately 10^{-2} . Noteworthy, the values of the optimized coefficients and the corresponding ranges of x have been listed in Table I and Table II.

Fig. 4 elaborates on the detailed working of the algorithm in the context of finding the optimal coefficients for (16).

TABLE I
OPTIMIZED COEFFICIENTS FOR $n = 3$

Range	α	β	ζ
x∈[0,0.04338)	0.2222	0.3889	0.3889
x∈[0.04338,0.1602)	0.1919	0.3578	0.3585
x∈[0.1602,0.2903)	0.1665	0.3332	0.3332
x∈[0.2903,0.6007)	0.1594	0.3299	0.3266
x∈[0.6007,0.7575)	0.1664	0.3333	0.3333
x∈[0.7575,1.135)	0.1703	0.3335	0.3359
x∈[1.135,1.335)	0.1667	0.3333	0.3333
x∈[1.335,1.795)	0.1643	0.3333	0.3322
x∈[1.795,1.959)	0.1667	0.3333	0.3333
x∈[1.959,2.096)	0.1675	0.3333	0.3335
x∈[2.096,2.646)	0.1693	0.3333	0.3336
x∈[2.646,2.887)	0.1669	0.3333	0.3333
x∈[2.887,3.654)	0.1624	0.3333	0.3331
x∈[3.654,5]	0.1285	0.3333	0.3332

Figs. 5a and 5b show the performance of the optimized coefficients for $n = 3$ and $n = 4$ in terms of the RE against the original Sadhwani−Yadav−Aggarwal bounds [17]. The 'sharp points' in the optimized plots basically indicate the change of optimization range owing to which the values assigned to the optimized coefficients changes and hence the path traced by the optimized approximation changes. It is evident that the optimized approximation outperforms the original bounds. Moreover as evident from Figs. 5a and 5b, the frequency of the 'dips' occurring in the RE plot for optimized approximation is high as compared to the original bounds. This indicates that the fluctuation in the RE is smaller in magnitude as well as shorter in the range of x; over which they occur as compared to the original bounds. Due to these characteristics, the optimized approximation has a

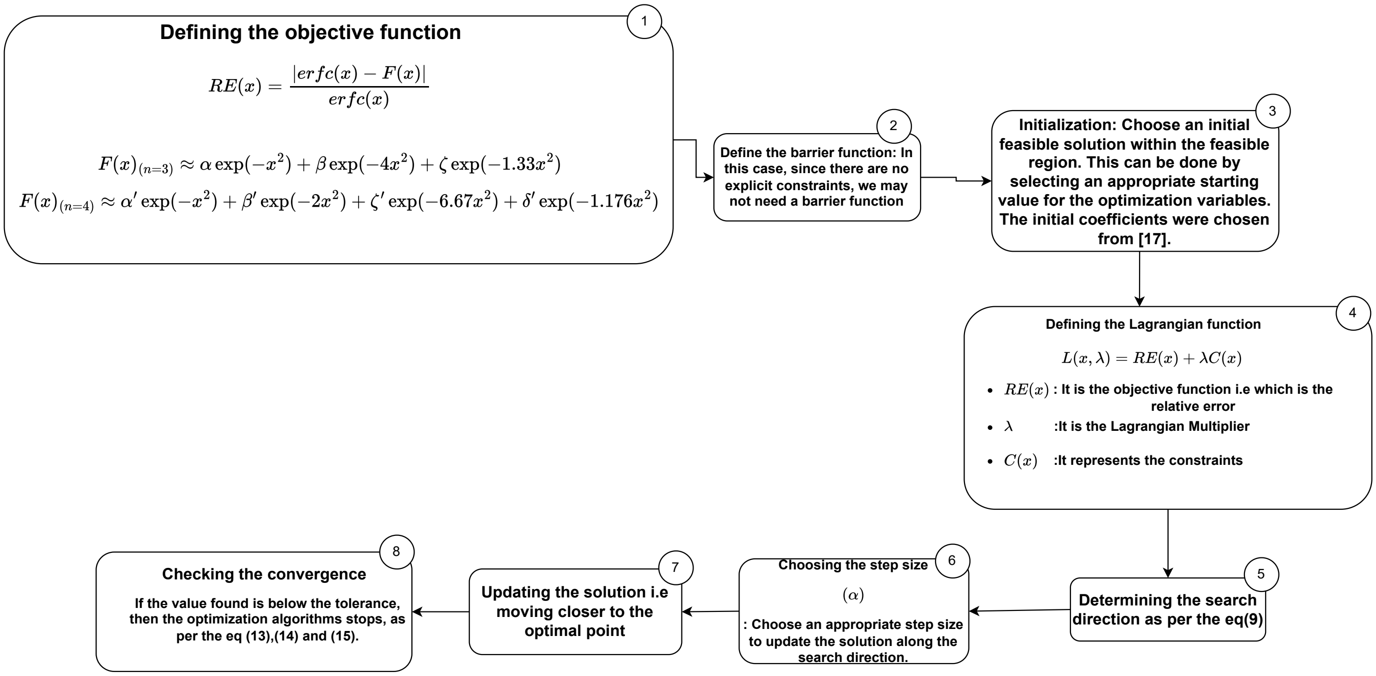
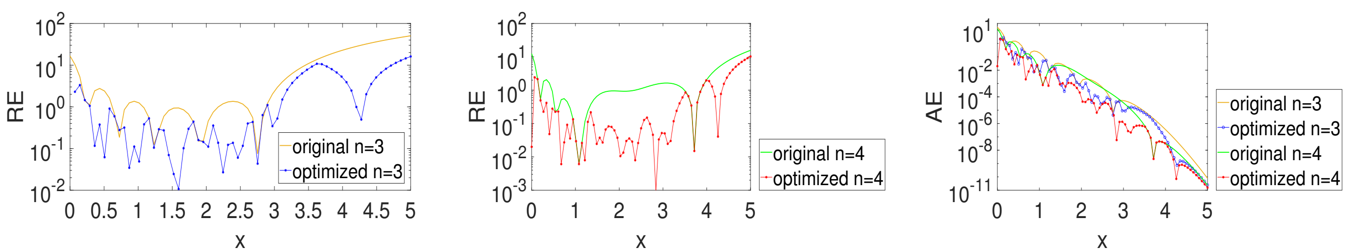


Fig. 4. Flow chart to find the optimized coefficients as per (16).



(a) (b) (c)

Fig. 5. Accuracy Comparison of the proposed optimized approximation (18) over the original (16) in terms of RE and AE.

TABLE II
OPTIMIZED COEFFICIENTS FOR $n = 4$

Range	α'	β'	ζ'	δ'
x∈[0,0.04338)	0.1562	0.2812	0.2812	0.2812
x∈[0.04338,0.1368)	0.1348	0.2597	0.2593	0.2598
x∈[0.1368,0.2102)	0.1250	0.2500	0.2500	0.2500
x∈[0.2102,0.4505)	0.1213	0.2465	0.2474	0.2464
x∈[0.4505,0.5306)	0.1235	0.2488	0.2496	0.2485
x∈[0.5306,0.6040)	0.1250	0.2500	0.2500	0.2500
x∈[0.6040,0.8343)	0.1263	0.2507	0.2500	0.2511
x∈[0.8343,0.9377)	0.1256	0.2503	0.2500	0.2505
x∈[0.9377,1.198)	0.1250	0.2500	0.2500	0.2500
x∈[1.198,1.340)	0.1253	0.2500	0.2500	0.2502
x∈[1.340,1.518)	0.1264	0.2501	0.2500	0.2509
x∈[1.518,2.614)	0.1269	0.2500	0.2500	0.2509
x∈[2.614,3.521)	0.1277	0.2500	0.2500	0.2505
x∈[3.521,4.016)	0.1250	0.2500	0.2500	0.2500
x∈[4.016,5]	0.1190	0.2500	0.2500	0.2497

better performance. Furthermore, for some ranges of x, the original bounds and the optimized approximations overlap at best. This signifies that for such ranges, the coefficients used in the original bounds are matching the optimized coefficients. This is also evident from Table I and Table II. Clearly, with the help of these optimized coefficients, the RE is minimal for the entire range considered over here i.e. $x \in [0, 5]$. Noteworthy, the coefficients listed in these Tables have a very generic use as for the same values of coefficients the AE is also getting optimized which can be seen from Fig. 5c.

A. Accuracy Comparison of (18)

In this section, the accuracy of (18) is demonstrated by presenting exhaustive graphical as well as numerical comparisons with several existing well-known approximations/bounds available in the open literature. This is achieved via RE as defined in (17).

In Fig. 6a the accuracy of (18a) is compared with the existing [23], [19], [20] for three exponential terms. Clearly, the optimized one has a better performance than all the other approximations. Although [19, $N = 3$] has an appreciable performance till $x \approx 3.25$, it is evident that the RE becomes unbounded just after x exceeds this value thereby limiting its significance. On the contrary, Eq. (18a) gives accurate results for the complete range of x under consideration with the RE

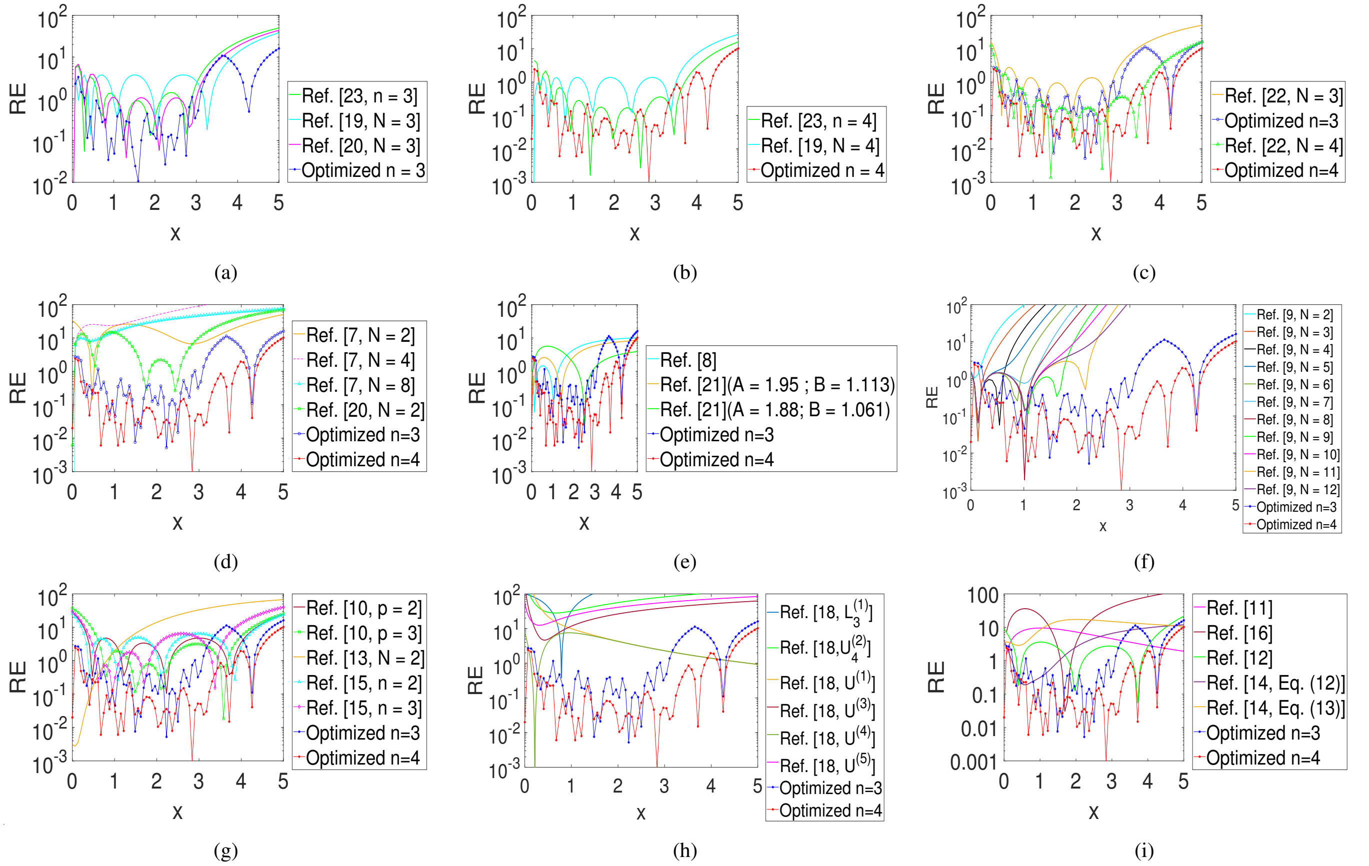


Fig. 6. Accuracy comparison of (18) with several existing bounds on the Gaussian Q function [7]–[16], [18]–[23]

reaching to as low as 10^{-2} that too with only three terms. Moreover, as seen from the plot, although the corresponding simple exponential based approximations [23] and [20] have an appreciable performance yet they are not as accurate as the optimized approximation (18a) which again highlights the utility of the proposed work.

In Fig. 6b, the accuracy of (18b) is compared with the corresponding approximations [19, $N = 4$] and [23, $N = 4$]. Apart from giving a very low value of the RE, the approximation [19, $N = 4$] has a behaviour similar to the above discussed $N = 3$ case of [19] i.e. it becomes unbounded after $x \approx 3.25$; unlike the proposed analysis where it is clearly seen that the optimized $n = 4$ has a better performance than the optimized $n = 3$. Furthermore, the approximation as described in [23] for $n = 4$ is superceded by the optimized (18b) for a quite a large range of x; barring a couple of points where [23] performs better. In Fig. 6c, an exponential based approximation given by the authors in [22] is compared with the proposed optimized approximation (18). It can bee seen that for like number of terms, Eq. (18) is more accurate than [22].

Fig. 6d demonstrates the RE comparison of (18) with the exponential bounds on the Gaussian Q function [7] and [20, $N = 2$] derived via numerical integration techniques. It is evident that (18) outperforms [7, $N = 2$] case. Moreover, to provide an insight on the effect of increasing the number of terms on the accuracy, the generic approximation as given in [7] is extended to $N = 4, 8$ terms. It can be seen that (18) is having a clear cut supremacy over both the aforementioned cases of [7] that too with less number of terms. Furthermore, the proposed optimized approximations completely supersede $N = 20$ of [20]; as evident from Fig. 6d.

In Fig. 6e, the RE comparison of (18) with [8] is demonstrated. Quite clearly, Eq. (18) surpasses this approximation which significantly looses its accuracy beyond $x \geq 1$. Furthermore, Eq. (18) is compared with an improved version of [8], as described in [21]. The authors in [21] have proposed two sets of the optimized coefficients for the original proposed approximation [8]. They are defined as: $[a = 0.3760, b = 0.5, c = 1.3293]$ and $[a = 0.3200, b = 0.4703, c = 1.5625]$. The former is used to minimize the RE whereas the latter reduces the AE as well as the total error. It should be noted that the latter is achieved at the cost of the unbounded RE. On the contrary, it is evident that the proposed optimized approximation (18) supersedes all these cases in terms of the RE without worrying about the AE or the total error (as the optimized coefficients here are generic in nature).

Fig. 6f further compares (18) with an $N-$term approximated version [9] of the original approximation [8]. The authors in [9] have clearly stated that with minimum eight terms (i.e. $N = 8$), the performance of the approximation [9] somewhat matches to [8]. However, when the analysis of [9] is further extended upto twelve terms; it becomes clear that even $N = 12$ case of [9] is not comparable to the proposed optimized approximation (18).

TABLE III
ACCURACY COMPARISON OF OPTIMIZED $n = 3$ ON THE BASIS OF MEAN RE (%)

Over range $x \in [0, 1.335)$							
Approximations	x∈[0,0.04338)	x∈[0.04338,0.1602)	x∈[0.1602,0.2903)	x∈[0.2903,0.6007)	x∈[0.6007,0.7575)	x∈[0.7575,1.135)	x∈[1.135,1.335)
Ref. [7, $N = 2$]	28.419190	25.667450	16.304275	5.007252	11.281199	20.663444	25.870674
Ref. [10, $p = 2$]	24.453878	21.142647	12.431306	3.094533	4.061264	3.595622	0.709423
Ref. [10, $p = 3$]	31.977425	30.118883	21.759821	10.474350	2.251471	1.348332	1.457680
Ref. [13, $N = 2$]	0.027457	0.026623	0.012338	0.125080	0.535857	1.692214	3.813088
Ref. [15, $n = 2$]	21.759106	17.979792	9.348845	2.477296	4.221611	1.972619	2.946880
Ref. [15, $n = 3$]	25.192627	22.115541	13.794443	4.359500	0.395063	0.51665	1.549789
Ref. [16]	2.163090	10.794937	21.481216	32.700186	34.846784	25.000860	11.486097
Ref. [18, $L_3^{(1)}$]	88.041183	90.781276	76.293396	48.579315	15.941595	23.944838	67.139025
Ref. [18, $U^{(1)}$]	∞	259.117285	75.940916	28.1462509	15.0562528	10.553858	7.958564
Ref. [18, $U^{(3)}$]	∞	33.715400	9.761728	5.0625113	6.860392	10.742490	15.340895
Ref. [11]	0.878976	2.344217	4.379186	7.028676	8.753572	9.304845	9.336795
Ref. [12]	7.422319	6.139602	3.265724	1.044461	2.560162	3.432892	3.310210
Ref. [20, $N = 2$]	2.063028	8.848748	12.767578	5.904998	9.287354	13.938057	9.970142
Ref. [20, $N = 3$]	1.918228	6.150275	3.770336	2.856944	1.400202	0.787615	0.341808
Ref. [19, $N = 3$]	1.672025	2.557369	2.522674	1.981657	3.596147	1.745932	2.193061
Ref. [22, $N = 3$]	13.169293	8.274451	1.601509	2.187704	0.546974	1.079513	0.299340
Ref. [23, $n = 3$]	1.896462	5.794870	2.949475	2.402900	0.574322	1.110750	0.277721
Original $n = 3$ [17]	13.169293	8.274451	1.601509	2.1877046	0.546974	1.079513	0.299340
Optimized $n = 3$	**2.128596**	**2.433114**	**1.620761**	**0.406501**	**0.529959**	**0.210912**	**0.299116**
Over range $x \in [1.335, 5]$							
Approximations	x∈[1.335,1.795)	x∈[1.795,1.959)	x∈[1.959,2.096)	x∈[2.096,2.646)	x∈[2.646,2.887)	x∈[2.887,3.654)	x∈[3.654,5]
Ref. [7, $N = 2$]	23.968946	18.959118	15.883771	10.547755	6.789599	9.949436	32.225438
Ref. [10, $p = 2$]	2.781886	2.753358	1.706414	1.649122	4.178571	3.714215	10.099779
Ref. [10, $p = 3$]	0.489817	0.717572	0.380263	1.246852	2.897781	2.030765	11.774554
Ref. [13, $N = 2$]	7.5146095	11.862786	14.190777	20.884772	28.336534	38.548317	57.307183
Ref. [15, $n = 2$]	4.851303	3.259557	1.541462	2.524127	5.575020	4.894135	9.933681
Ref. [15, $n = 3$]	1.071317	1.534044	3.025767	5.585689	6.021592	2.935163	21.890487
Ref. [16]	2.012226	1.159972	3.826488	14.735933	29.723376	51.193177	96.533039
Ref. [18, $L_3^{(1)}$]	114.83067	158.7371	178.4973	232.88089	287.01038	3.61211e+02	5.1077e+02
Ref. [18, $U^{(1)}$]	5.991942	4.641747	4.123123	3.379301	2.606680	2.004161	1.232963
Ref. [18, $U^{(3)}$]	20.060348	24.315903	26.139460	31.522316	36.402051	42.998235	55.215958
Ref. [11]	8.488743	7.515292	6.987314	6.154914	5.045383	4.021853	2.537793
Ref. [12]	2.032786	0.532736	0.225748	1.612643	2.610579	2.0250664	8.246632
Ref. [20, $N = 2$]	2.967372	1.382184	2.082375	1.419074	5.032805	17.924599	50.249420
Ref. [20, $N = 3$]	0.826981	0.719260	0.223196	0.764118	0.477976	4.626844	26.581016
Ref. [19, $N = 3$]	3.344444	1.634577	0.372011	2.537864	3.587209	2.318335	21.081197
Ref. [22, $N = 3$]	0.769579	0.223809	0.436818	1.139260	0.465444	6.898076	31.837122
Ref. [23, $n = 3$]	0.735050	0.205563	0.466186	1.199597	0.465712	6.796889	31.651446
Original $n = 3$ [17]	0.769579	0.213809	0.436818	1.139260	0.465444	6.898076	31.837128
Optimized $n = 3$	**0.152326**	**0.214889**	**0.157080**	**0.161006**	**0.470013**	**4.277285**	**5.766500**

In Figs. 6g, 6h and 6i, the analysis is further solidified by comparing (18) with the remaining significant approximations [10]–[16], [18]; eventually super ceding all of them. Noteworthy, on the basis of mean RE for each range of x, Table III and Table IV provides an exhaustive numerical comparison of (18) with all the so far discussed approximations/bounds. It is quite evident that for almost each range of x, Eq. (18) gives minimum mean RE. In addition, the total mean RE of (18) is also the lowest among all the approximations as evident in Table V. This gives an idea of the significance of the proposed analysis.

B. Applications of (18)

The new optimized coefficients of (18) are applicable as one to one replacements in the applications of the original one (16) like:

- the error analysis under $\kappa - \mu$ shadowed fading channel [32]
- radio-on-free-space optical systems [33]
- terrestrial free space optical systems [34]
- frequency-modulated differential chaos shift keying ultra-wide band system [35]
- re-configurable intelligent surfaces−aided wireless communication systems [36]–[38]
- bit error computation of M-Ary phase-shift keying signals [39]
- cognitive radio-inspired non-orthogonal multiple access (CR-NOMA) systems [40]

IV. CONCLUSION AND FUTURE SCOPE

This paper proposes a generic method which optimizes already existing approximations of the Gaussian Q function

TABLE IV
ACCURACY COMPARISON OF OPTIMIZED $n = 4$ ON THE BASIS OF MEAN RE (%)

for range $x \in [0, 0.9377)$

Approximations	x∈[0,0.04338)	x∈[0.04338,0.1368)	x∈[0.1368,0.2102)	x∈[0.2102,0.4505)	x∈[0.4505,0.5306)	x∈[0.5306,0.6040)	x∈[0.6040,0.8343)	x∈[0.8343,0.9377)
Ref. [7, $N = 4$]	2.133276	8.748786	15.774615	23.012107	24.929919	24.981012	23.935755	22.783481
Ref. [7, $N = 8$]	1.999793	6.893963	9.707369	8.900959	8.071891	8.725058	10.637198	12.826009
Ref. [8]	1.084637	0.486164	0.255595	1.082157	1.421014	1.433479	1.1740930	0.648444
Ref.[3] [21]	0.730513	0.285371	0.917634	1.961181	2.460767	2.564386	2.444593	2.012931
Ref.[4] [21]	0.458997	1.422596	2.622710	4.174169	5.063256	5.391175	5.615989	5.418772
Ref. [18, $U_4^{(2)}$]	∞	109.9567	62.859238	38.837448	29.005497	28.118698	28.169996	29.089423
Ref. [18, $U^{(4)}$]	4.619474	2.296793	0.281397	1.241480	3.799970	5.050982	6.626345	7.257248
Ref. [18, $U^{(5)}$]	30.447251	21.243087	14.716010	12.755277	13.682856	15.082504	17.911543	20.584777
Ref. [9, $N = 8$]	1.165034	0.490577	0.252049	1.078487	1.417709	1.429627	1.161899	0.601551
Ref. [9, $N = 12$]	1.165034	0.490577	0.252049	1.078518	1.418039	1.430721	1.171777	0.646592
Ref. [19, $N = 4$]	0.860263	0.886415	0.512032	0.846677	0.850777	1.310359	0.767435	0.535416
Ref. [14, Eq. (12)]	3.155174	2.209662	1.314807	0.511851	0.203756	0.186167	0.214067	0.260228
Ref. [14, Eq. (13)]	3.232186	3.440770	3.361248	2.971744	2.900368	3.257371	4.562455	6.448614
Ref. [22, $N = 4$]	9.382129	4.855581	0.848701	1.387901	0.375407	0.266378	0.569106	0.161937
Ref. [23, $n = 4$]	1.773730	4.040680	1.894493	1.501626	0.352959	0.367788	0.576107	0.1466766
Original $n = 4$ [17]	9.379664	4.820683	0.824250	1.680550	0.710275	0.150319	0.473985	0.263023
Optimized $n = 4$	**2.0758060**	**1.767056**	**0.824250**	**0.307999**	**0.234394**	**0.150318**	**0.065325**	**0.067613**

for range $x \in [0.9377, 5]$

Approximation	x∈[0.9377,1.198)	x∈[1.198,1.340)	x∈[1.340,1.518)	x∈[1.518,2.614)	x∈[2.614,3.521)	x∈[3.521,4.016)	x∈[4.016,5]
Ref. [7, $N = 4$]	24.532184	27.668350	31.904530	51.955089	94.403614	130.9335	1.727154e+02
Ref. [7, $N = 8$]	16.060139	19.489454	22.738839	32.976327	47.481502	57.497191	68.217277
Ref. [8]	0.309971	1.057356	1.880834	4.697947	7.763991	9.012922	9.806484
Ref.[5] [21]	1.378744	0.516867	0.271758	2.934814	5.968775	7.223284	8.027049
Ref.[6] [21]	5.099015	4.453692	3.940885	1.538369	1.431159	2.701110	3.528793
Ref. [18, $U_4^{(2)}$]	32.708266	36.818029	41.434645	57.854965	83.451002	100.41395	1.1716e+02
Ref. [18, $U^{(4)}$]	7.270951	6.704147	6.247863	4.183397	2.241965	1.554463	1.133450
Ref. [18, $U^{(5)}$]	24.266808	27.802477	31.251320	41.989518	57.615332	67.720729	77.641077
Ref. [9, $N = 8$]	0.484157	2.013527	4.583191	1.1670e+02	2.18948e+03	1.05758e+04	5.20502e+04
Ref. [9, $N = 12$]	0.310374	1.060918	1.893119	8.391099	2.59175e+02	2.33557e+03	2.61050e+04
Ref. [19, $N = 4$]	1.244444	0.955910	0.281667	0.832076	0.959471	4.060296	16.592964
Ref. [14, Eq. (12)]	0.350543	0.521653	0.790233	2.888587	6.754776	9.001510	10.851682
Ref. [14, Eq. (13)]	9.240391	12.104564	14.297176	16.658844	14.77476	13.006591	11.539459
Ref. [22, $N = 4$]	0.218068	0.191862	0.057200	0.117943	0.215133	1.452187	8.839423
Ref. [23, $n = 4$]	0.210540	0.188360	0.057586	0.118684	0.213645	1.457515	8.846600
Original $n = 4$ [17]	0.040006	0.207073	0.534460	0.959766	1.440346	0.823038	8.239151
Optimized $n = 4$	**0.040006**	**0.092851**	**0.09730**	**0.048096**	**0.175077**	**0.823038**	**3.886734**

[1] $A = 1.95, B = 1.113$; [2] $A = 1.88, B = 1.061$

TABLE V
TOTAL MEAN RE COMPARISON OVER $x \in [0, 5]$

comparison with optimized $n = 3$		comparison with optimized $n = 4$	
Approximation	Total MRE	Approximation	Total MRE
Ref. [7, $N = 2$]	19.624517	Ref. [7, $N = 4$]	82.732208
Ref. [10, $p = 2$]	5.780641	Ref. [7, $N = 8$]	38.988705
Ref. [10, $p = 3$]	6.351980	Ref. [8]	5.570858
Ref. [13, $N = 2$]	26.845746	Ref.[7] [21]	4.480218
Ref. [15, $n = 2$]	6.033053	Ref.[8] [21]	2.913968
Ref. [15, $n = 3$]	9.032484	Ref. [18, $U_4^{(2)}$]	∞
Ref. [16]	43.674612	Ref. [18, $U^{(4)}$]	3.266679
Ref. [18, $L_3^{(1)}$]	2.6683e+02	Ref. [18, $U^{(5)}$]	48.229511
Ref. [18, $U^{(1)}$]	∞	Ref. [9, $N = 8$]	1.17554e+04
Ref. [18, $U^{(3)}$]	∞	Ref. [9, $N = 12$]	5.44124e+03
Ref. [11]	5.434935	Ref. [19, $N = 4$]	4.295977
Ref. [12]	3.903917	Ref. [14, Eq. (12)]	5.095305
Ref. [20, $N = 2$]	19.787045	Ref. [14, Eq. (13)]	11.988856
Ref. [20, $N = 3$]	8.665695	Ref. [22, $N = 4$]	2.317322
Ref. [19, $N = 3$]	7.480283	Ref. [23, $n = 4$]	2.216729
Ref. [22, $N = 3$]	10.514913	Original $n = 4$ [17]	2.535685
Ref. [23, $n = 3$]	10.332343	**Optimized** $n = 4$	**0.987178**
Original $n = 3$ [17]	10.514913	-	-
Optimized $n = 3$	**2.837328**	-	-

[1] $A = 1.95, B = 1.113$; [2] $A = 1.88, B = 1.061$

yielding new optimized coefficients giving rise to extremely accurate optimized approximations. To show the utility of the proposed approach, as an example, the range wise optimization of [17] is extensively presented in this paper. Exhaustive graphical as well as numerical comparisons have been carried out yielding the minimum mean RE not only for the smaller intervals of x; but also for the entire range of x under consideration. This paper also provides an insight on the usefulness of the optimized approximation in the various cutting-edge applications of the communication systems like [32]–[40].

As a future work, the range-wise coefficients can be used in the scenarios where the effect of noise is more i.e. the signal-to-noise ratio (SNR) is very low and we need accurate estimation of the performance analysis metrics. Moreover, the proposed coefficients can also be used in accurate computation of the symbol error probability of various digital modulation schemes subjected to severe fading conditions. Lastly, other simple exponential-based approximations [17], [19], [20], [23], [24], [41] can also be optimized using the proposed approach.

REFERENCES

[1] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels*, 2nd ed. Wiley, 2005.

[2] M. Ramirez and S. Molano, "Analytical expression for the bit error rate of a filtered multitone modulation scheme (FMT) through a multipath

fading channel," *IEEE Latin America Transactions*, vol. 19, no. 4, pp. 643–651, Apr. 2021.

[3] S. Aggarwal, "Approximate solution to SEP integral over Fluctuating Beckmann fading," *Wireless Personal Communications*, vol. 111, pp. 1367–1376, 2020.

[4] D. N. and R. B., "Performance analysis of wireless powered sensor network with opportunistic scheduling over generalized $\kappa-\mu$ shadowed fading channels," *Physical Communication*, vol. 53, p. 101727, 2022.

[5] M. Bilim, "Cascaded double fading channels: A case of $\kappa-\mu$ shadowing," *Physical Communication*, vol. 52, p. 101649, 2022.

[6] G. Llano Ramirez, J. Reig, and L. Rubio, "Modeling and analysis in frequency of the UWB channel with lognormal statistics for MB-OFDM systems," *IEEE Latin America Transactions*, vol. 7, no. 1, pp. 33–41, Mar. 2009.

[7] M. Chiani, D. Dardari, and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 4, pp. 840–845, Jul. 2003.

[8] G. K. Karagiannidis and A. S. Lioumpas, "An improved approximation for the Gaussian Q-function," *IEEE Commun. Lett.*, vol. 11, no. 8, pp. 644–646, Aug. 2007.

[9] Y. Isukapalli and B. Rao, "An analytically tractable approximation for the Gaussian Q-function," *IEEE Commun. Lett.*, vol. 12, no. 9, pp. 669–671, Sep. 2008.

[10] P. Loskot and N. C. Beaulieu, "Prony and polynomial approximations for evaluation of the average probability of error over slow-fading channels," *IEEE Trans. Veh. Technol.*, vol. 58, no. 3, pp. 1269–1280, Mar. 2009.

[11] W. M. Jang, "A simple upper bound of the Gaussian Q-function with closed-form error bound," *IEEE Commun. Lett.*, vol. 15, no. 2, pp. 157–159, Feb. 2011.

[12] M. López-Benítez and F. Casadevall, "Versatile, accurate, and analytically tractable approximation for the Gaussian Q-function," *IEEE Trans. Commun.*, vol. 59, no. 4, pp. 917–922, Apr. 2011.

[13] Q. Shi and Y. Karasawa, "An accurate and efficient approximation to the Gaussian Q-function and its applications in performance analysis in Nakagami-m fading," *IEEE Commun. Lett.*, vol. 15, no. 5, pp. 479–481, May 2011.

[14] G. Abreu, "Very simple tight bounds on the Q-function," *IEEE Trans. Commun.*, vol. 60, no. 9, pp. 2415–2420, Sep. 2012.

[15] O. Olabiyi and A. Annamalai, "Invertible exponential-type approximations for the Gaussian probability integral $Q(x)$ with applications," *IEEE Wireless Commun. Lett.*, vol. 1, no. 5, pp. 544–547, Oct. 2012.

[16] Y. Mouchtak and F. E. Bouanani, "New tighter upper bounds on the performance of convolutional code over exponentially correlated Rayleigh fading channel," in *Proc. IEEE International Wireless Commun. and Mobile Computing Conf. (IWCMC)*, Sep. 2016.

[17] D. Sadhwani, R. N. Yadav, and S. Aggarwal, "Tighter bounds on the Gaussian Q function and its application in Nakagami$-m$ fading channel," *IEEE Wireless Commun. Lett.*, vol. 6, no. 5, pp. 574–577, Oct. 2017.

[18] F. E. Bouanani, Y. Mouchtak, and G. K. Karagiannidis, "New tight bounds for the Gaussian Q-function and applications," *IEEE Access*, vol. 8, pp. 145 037–145 055, Aug. 2020.

[19] I. M. Tanash and T. Riihonen, "Global minimax approximations and bounds for the Gaussian Q-function by sums of exponentials," *IEEE Trans. Commun.*, vol. 68, no. 10, pp. 6514–6524, Oct. 2020.

[20] ——, "Quadrature-based exponential-type approximations for the Gaussian Q-function," in *Proc. IEEE 93rd Vehicular Technol. Conf. (VTC2021-Spring)*, Apr. 2021.

[21] ——, "Improved coefficients for the Karagiannidis–Lioumpas approximations and bounds to the Gaussian Q-function," *IEEE Commun. Lett.*, vol. 25, no. 5, pp. 1468–1471, May 2021.

[22] S. Aggarwal, "Average of product of two Gaussian Q functions as summation series and its significance in evaluating SEP integrals over fading channels," *Journal of the Franklin Institute*, vol. 138, no. 15, pp. 8045–8060, Oct. 2021.

[23] D. Sadhwani, A. Powari, and N. Mehta, "New, simple and accurate approximation for the Gaussian Q function with applications," *IEEE Commun. Lett.*, vol. 26, no. 3, pp. 518–522, Mar. 2022.

[24] A. Powari and D. Sadhwani, "Extremely accurate, generic and tractable exponential bounds for the Gaussian Q function with applications in communication systems," *IEEE Commun. Lett.*, vol. 27, no. 7, pp. 1724–1728, Jul. 2023.

[25] G. Astfalk, I. Lustig, R. Marsten, and D. Shanno, "The interior-point method for linear programming," *IEEE Software*, vol. 9, no. 4, pp. 61–68, 1992.

[26] S.-J. Kim, K. Koh, M. Lustig, S. Boyd, and D. Gorinevsky, "An interior-point method for large-scale ℓ_1-regularized least squares," *IEEE Journal of Selected Topics in Signal Processing*, vol. 1, no. 4, pp. 606–617, 2007.

[27] C.-N. Lu and M. Unum, "Network constrained security control using an interior point algorithm," *IEEE Transactions on Power Systems*, vol. 8, no. 3, pp. 1068–1076, 1993.

[28] J. Momoh and J. Zhu, "Improved interior point method for OPF problems," *IEEE Transactions on Power Systems*, vol. 14, no. 3, pp. 1114–1120, 1999.

[29] P. Zhao and S. Yu, "An improved interior point algorithm for quantile regression," *IEEE Access*, vol. 8, pp. 139 647–139 657, 2020.

[30] M. I. Khan and K. Xia, "Effective self adaptive multiple source localization technique by primal dual interior point method in binary sensor networks," *IEEE Communications Letters*, vol. 21, no. 5, pp. 1119–1122, 2017.

[31] A. Powari, D. Sadhwani, L. Gupta, and R. N. Yadav, "Optimization of the exponential bounds on the Gaussian Q function using interior point algorithm and its application in communication theory," *IEEE Open Journal of the Communications Society*, pp. 1–1, 2022.

[32] D. Sadhwani, "Simple and tightly approximated integrals over κ-μ shadowed fading channel with applications," *IEEE Trans. Veh. Technol.*, vol. 67, no. 10, pp. 10 092–10 096, 2018.

[33] M. P. Ninos, H. E. Nistazakis, H. G. Sandalidis, and G. S. Tombras, "CDMA RoFSO links with nonzero boresight pointing errors over M turbulence channels," *IEEE Photonics Journal*, vol. 10, no. 5, pp. 1–12, Dec. 2018.

[34] G. K. Varotsos, H. E. Nistazakis, H. Sandalidis, G. S. Tombras, and W. Gappmair, "SIMO subcarrier PSK FSO links with phase noise and non-zero boresight pointing errors over turbulence channels," *IET Communications*, vol. 13, no. 7, pp. 831–836, 2019.

[35] H. Ma, G. Cai, Y. Fang, J. Wen, P. Chen, and S. Akhtar, "A new enhanced energy-detector-based FM-DCSK UWB system for tactile internet," *IEEE Transactions on Industrial Informatics*, vol. 15, no. 5, pp. 3028–3039, 2019.

[36] M. H. Khoshafa, T. M. N. Ngatched, M. H. Ahmed, and A. R. Ndjiongue, "Active reconfigurable intelligent surfaces-aided wireless communication system," *IEEE Commun. Lett.*, vol. 25, no. 11, pp. 3699–3703, Dec. 2021.

[37] L. Yang, J. Yang, W. Xie, M. O. Hasna, T. Tsiftsis, and M. D. Renzo, "Secrecy performance analysis of RIS-aided wireless communication systems," *IEEE Trans. Veh. Technol.*, vol. 69, no. 10, pp. 12 296–12 300, 2020.

[38] P. Yang, L. Yang, W. C. Kuang, and S. J. Wang, "Outage performance of cognitive radio networks with a coverage-limited RIS for interference elimination," *IEEE Wireless Commun. Lett.*, pp. 1–1, 2021.

[39] Y. Jia, Z. Wang, J. Yu, and P.-Y. Kam, "A new approach to deriving closed-form bit error probability expressions of MPSK signals," *IEEE Trans. Commun.*, pp. 1–1, 2023.

[40] H. Lei, F. Yang, I. S. Ansari, H. Liu, K. J. Kim, and T. A. Tsiftsis, "Secrecy outage performance analysis for uplink CR-NOMA systems with hybrid SIC," *IEEE Internet of Things Journal*, pp. 1–1, 2023.

[41] D. Sadhwani and R. N. Yadav, "A comprehensible form of the product of two Gaussian Q functions and its usefulness in κ-μ shadowed fading distribution," *Journal of the Franklin Institute, Elsevier*, vol. 357, pp. 5110–5123, Apr. 2020.

ADITYA POWARI is pursuing MSc in Communications and Signal Processing at the Department of Electrical and Electronic Engineering, The University of Manchester, U.K. He is also working as a research assistant in the Department of Engineering, University of Cambridge, UK. He has previously worked as a research intern at the Department of Electrical and Computer Engineering, NUS Singapore and at Department of Electrical Engineering, IIT, Delhi. His research interests span over the fields of Wireless Communications, NOMA for 6G systems, Molecular Communications, Information Theory, Internet of Everything (IoE) and Digital Signal Processing.

GARV ANAND is pursuing his B.Tech student in Electrical Engineering with minors in Computer Science from Institute of Infrastructure, Technology, Research and Management, Ahmedabad, India. He has served as a research intern at IIITDM Kancheepuram, India, and NSF-TIH-IIT Bombay in collaboration with the University of Colorado Boulder. With a keen focus on Data Analysis, Machine Learning(ML), Digital Signal Processing, and the Internet of Things (IoT), he has a keen interest in making digital communication more sustainable, striving to bridge technology with eco-friendly solutions.

DHARMENDRA SADHWANI is working as an Assistant Professor in the Department of Electrical and Computer Science Engineering, Institute of Infrastructure Technology Research and Management, Ahmedabad, India. His research interests include the application of Gaussian Q function in communication theory, error performance analysis and identification of more efficient higher order digital modulation schemes, statistical characterization of fading channels, satellite communication systems, 5G communications, cooperative communication systems, under water acoustics (systems), poisson's cellular networks, body to body communications, shadowing in communication systems and application of machine learning/neural networks in communication theory.