The Probabilistic Behavior of the Set and Reset Thresholds in Knowm's SDC Memristors: Characterization and Simulation

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***Abstract*—This paper presents a proposal for the characterization of the set and reset thresholds for Knowm's SDC memristors. The purpose is to incorporate the variability of the hysteresis cycles within the Generalized Mean Metastable Switch (GMMS) memristor model and, in this way, be able to perform simulations that reproduce these phenomena in a meaningful and computationally efficient way. We depart from the assumption that their probabilistic behavior can be well represented by using α-stable random variables. The main advantage of using α-stable variables is that they can capture both skewness and high variability (i.e., heavy tails), which can be exhibited by the observed phenomenon. At the same time, they also include the Gaussian random variable as a particular case, thus increasing the modeling flexibility.**

***Index Terms*—α-stable random variable characterization, SDC memristor modeling, SDC memristor simulating.**

I. Introduction

The memristor, which is the fourth fundamental passive circuit element, was originally conceived by Leon Chua in the 1970s [1] being a device that presents a hysteresis cycle for its current as a function of the potential applied on its two electrodes and, therefore, incorporates some memory function. In the last two decades, since it was obtained as a solid-state component in HP laboratories in 2008 [2], it has generated great expectations as a promising candidate for the practical realization of new computing paradigms supported by computational operations that are essentially performed directly in memory such as neuromorphic computing [3]–[6].

Currently, there are some technologies for manufacturing memristors, which have reached a certain degree of maturity. The most popular technologies are the resistive-switching random access memory (ReRAM) [7] and the phase-change memory (PCM) [8]. In addition, there are devices manufactured by using the self-directed channel (SDC) technique. This kind of memristors is widely available and constitute a subclass of the electrochemical metallization (ECM) devices [9], [10].

Commercially available devices let us experiment with practical applications of the memory implicitly present in their hysteresis cycle. However, one striking feature in the physical operation of memristors, available for experimentation, is the evident variability of the hysteresis cycles around the thresholds for transiting between the set and reset states [11]–[14]. It should be remembered that the manufacturing process of a memristor device is not a simple matter, it starts with the construction of conductive channels by mobilizing ions, i.e. atoms, within certain crystalline substrates, which in principle involves, among other factors of a practical nature, the operating temperature, certain inertia phenomena and the foreseeable influence of quantum mechanical phenomena.

This variability in memristor hysteresis cycles involves unavoidable technical challenges when using such components in real circuits and it is no longer realistic to assume that the memristors exhibit instantaneous, homogeneous, and predictable responses, at least not for the time being. In fact, much of the current work is focused on exploiting the stochastic behavior of memristors for certain applications in computing [15], [16].

In this paper, we will employ a set of measurements obtained with commercial W-SDC (tungsten-doped SDC, manufactured by Knowm) memristors to show the usefulness of the proposed methodology in characterizing the variability of switching thresholds. It is worth emphasizing that the methodology that we explain herein can be applied to any other SDC variant and, more importantly, to any other type of memristor technology.

For a comprehensive overview of memristor modeling, with emphasis on the variability phenomenon, the reader is referred to [13] and references therein contained. In general, there are two approaches for modeling memristor variability. The first one addresses the phenomenon as a stochastic process and the second one as a chaotic phenomenon. Among the works that approach variability as a stochastic phenomenon, we have the ones that assume stationary processes [17]–[19], the ones that employ multivariate series [20], and more elaborate models that recognize a non-stationary behavior [21], [22]. In addition, within the pieces of research that approach variability as a chaotic phenomenon, we have the ones that ponder the convenience of a deterministic model [23] and even those that see in memristor variability a means to model and build chaotic circuits [24].

Specifically, regarding the characterization of SDC memristors, the accessibility of Knowm's devices has encouraged multiple research groups to experiment with them in an attempt to obtain their characterization [25], [26]. It is worth

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TABLE I
RELATED WORK

Ref.	Type	Contribution	Technology	Reports memristor variability
[1]	Seminal	Theoretical	NA	NA
[2]	Seminal	Invention/Discovery	TiO_2	NA
[3]	News	NA	Several	Yes
[4]	R&D	Theoretical	Generic	No
[5]	Review	Topic review	Emergent	No
[6]	R&D	Invention/Discovery	Nanotech.	No
[7]	Seminal	Invention/Discovery	ReRAM TiON	Implicit
[8]	Seminal	Invention/Discovery	PCM	Yes
[9]	Seminal	Invention/Discovery	ECM/SDC	Yes
[10]	R&D	Invention/Discovery	ECM/SDC	Yes
[11]	R&D	Characterization/ Modeling	ReRAM	Yes
[12]	R&D	Characterization/ Modeling	ReRAM	Yes
[13]	Survey	Review	Several	Yes
[14]	Thecnical	Specs	SDC	Yes
[15]	R&D	Applications	Several	Yes
[16]	R&D	Applications	Several	Yes
[17]	R&D	Characterization/ Modeling	ReRAM TiO_2	Yes
[18]	R&D	Modeling	Several	Yes
[19]	R&D	Modeling	ReRAM	Yes
[20]	R&D	Characterization/ Modeling	ReRAM HfO_2	Yes
[21]	R&D	Characterization/ Modeling	ReRAM $ZrO_2(Y)$ TaO_x	Yes
[22]	R&D	Modeling	ReRAM	Yes
[23]	R&D	Characterization/ Modeling	SDC	Yes
[24]	R&D	Application	ReRAM emulator	Yes
[25]	R&D	Modeling	Generic & SDC	Implicit
[26]	R&D	Characterization	SDC	Yes
[27]	R&D	Modeling	SDC	No
[28]	R&D	Characterization/ Modeling	SDC	Yes

pointing out that the SDC memristor model developed by Knowm researchers provides an excellent starting point to incorporate improvements and make it more realistic and in agreement with experimental measurements and observations.

The model proposed by Knowm is the so called Generalized Mean Metastable Switch (GMMS) memristor model. It captures switching dynamics of the memristors by using a probabilistic approach [27]. There is also a study comparing the VTEAM and Strukov models applied to the case of Knowm memristors [28], as well as the specific proposal of Ostrovskii's team, who proposes adjustments to the GMMS model to generate hysteresis profiles with snapbacks and some chaotic variability [23]. Table I provides a brief comparison among pieces of related work. It is worth mentioning that none of these works makes use of α-stable distributions as a modeling approach.

In this work we propose to make use of α-stable random variables in order to characterize the variability of set and reset thresholds of Knowm's SDC memristors. Specifically, it is proposed to start from the hypothesis that α-stable random variables properly characterize the observed distribution profiles in the sample data. To test this assumption, a data set comprising a large number of threshold measurements is used to estimate the distribution parameters of α-stable random variables. The resulting model is intended to be incorporated in the GMMS model improved by Ostrovskii's team (here denoted as GMMS-Os model) [23]. Additionally, to allow interested researchers to test and eventually improve our characterization and simulation proposal, a demonstration code is provided in a GitHub repository [29].

In the remaining sections we will focus on the presentation and evaluation of our proposal. In Section II, the operation of SDC memristors is briefly described; in Section III, the model adopted in this work for the simulation work is described; in Section IV, some basic notions about the α-stable random variable are provided. In section V, we briefly explain the methodology used to characterize the variability of the switching thresholds using α-stable random variables. These explanations make references to the pieces of software that accompany this paper so that the interested reader can replicate our methodology and results. In Section VI, the results of the empirical characterization for the set and reset thresholds of a typical W-SDC memristor are presented; in Section VII, as a demonstrative example, we show the results of incorporating the experimentally characterized random variables in the GMMS-Os model. Finally, in Section VIII, we present our conclusions.

II. OVERVIEW OF KNOWM'S SDC MEMRISTORS

Knowm's SDC memristors do not operate like the ones generally known as Resistive RAM (ReRAM). That is, they are not based on the construction of a conductive filament between two electrodes, but rather on the creation of clusters of silver ions, within a conduction channel, whose density and distance among them determine the net resistivity when a potential difference is applied between its two electrodes. These agglomerations are caused by the initial mobilization of tin ions towards the interior of the active layer of the memristor, from an initial chemical reaction that creates fissures inside the crystal, allowing the penetration of tin ions which, in turn, facilitate the accumulation of silver ions which, in addition, tend to group together to form clusters. According to the information provided by the manufacturer [30], an SDC memristor consists of two tungsten electrodes and several layers (See Fig. 1, left) that include pure Ge2Se3 substrates, a silver layer (source layer), a SnSe layer (assist layer) and an active Ge2Se3 layer (active layer) doped with one of the elements used in the process (i.e., W, Sn, Cr or C), which determines the specific operation of the memristor.

The assist layer is the source of the tin ions (+Sn), which are intended to facilitate the agglomeration of the silver ions (+Ag), coming from the source layer to the active layer (see Fig. 1, right). The source layer is closer to one of the electrodes, whereas the active layer is closer to the other one.

When the potential at the electrode closer to the source layer is higher than the potential of the electrode closer to the active layer (so that a potential difference of the former minus the latter is positive), the memristor transits to its low resistance

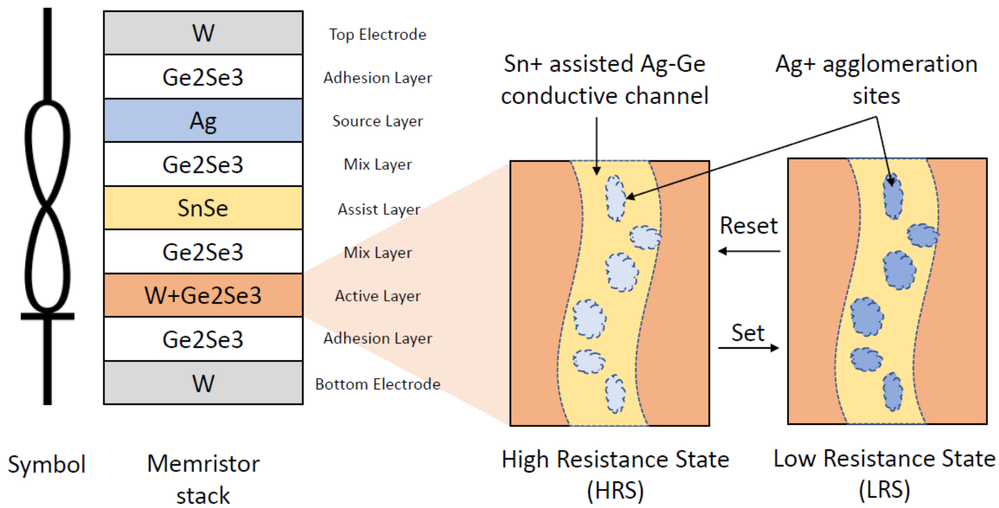

Fig. 1. Symbol and construction sketch for a Knowm W-SDC memristor. The symbol used here for the memristor device corresponds to the one used by Knowm's own documentation. We also follow Knowm's convention which considers that when the bar of the symbol is connected to the lower potential, the device will be driven to its high conductivity state.

state (LRS) and we have a set operation. In contrast, when such potential difference is sufficiently negative, then the memristor transits to a high resistance state (HRS) and we have a reset operation. For the sake of simplicity let us denote LRS and HRS as ON and OFF states, respectively.

For example, for the case of SDC memristors with doping made with W, Sn, or C, the forward threshold voltage (for set operation) is specified with a typical value of V_{ON}=+0.26 V, whereas the reverse threshold voltage (for reset operation) is specified at V_{OFF}=-0.11 V. When the device is driven between the ON and OFF states, its current follows a hysteresis curve whose distinguishing mark is a crossing point, around the change of the polarity sign (either from positive to negative or vice versa), which is known as the "pinch" in the hysteresis loop. This is illustrated in Fig. 2, where 1000-cycle hysteresis plots for a typical Knowm W-SDC type memristor are presented. They were obtained by using the test circuit shown in Fig. 3 with a test signal V of a triangular shape varying in the interval ±0.4 V with a frequency of 1 Hz. Note the variability in the various hysteresis plots, particularly with respect to the set and reset thresholds.

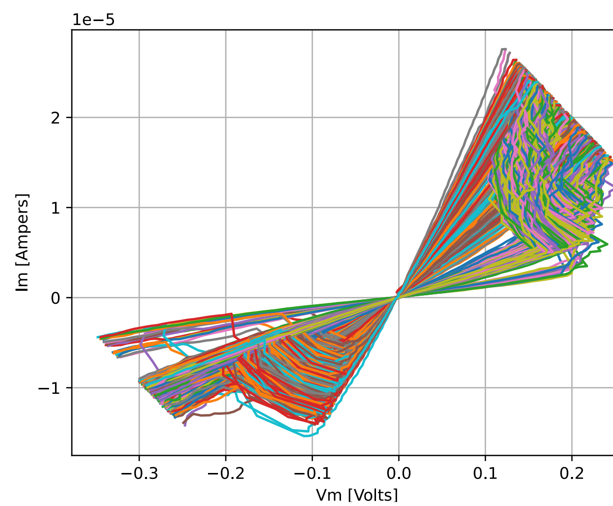

Fig. 2. I-V plots for 1000 hysteresis cycles of a typical Knowm W-SDC memristor.

III. Reference Model for Knowm Memristors

Knowm has developed some models for its memristors [27] departing from what is known as the Mean Metastable Switch (MMS) memristor model. This model was obtained in a semiempirical form and it describes the behavior of the memristor's conductive channel which, for modeling purposes, is considered to be composed of N metastable switching (MSS) elements that probabilistically switch between LRS and HRS as a function of the voltage bias and temperature. The model assumes a state variable, denoted by X, which represents the fraction of switches that are in the ON state. That is, this variable represents the number of switches in the ON state, which is scaled to take on values between 0 and 1. Thus, the change in X is defined by

$$dX = N_{OFF\to ON} - N_{ON\to OFF}, \tag{1}$$

where $N_{OFF\to ON} = P_{OFF\to ON}(1-X)$ and $N_{ON\to OFF} = P_{ON\to OFF}X$. In turn, the switching probabilities $P_{OFF\to ON}$ y $P_{ON\to OFF}$ depend on the applied potential difference V_m (see Fig. 3) and are defined by

$$P_{OFF\to ON} = \rho\Big[\frac{1}{1+e^{-\varepsilon(V_m - V_{ON})}}\Big] \tag{2}$$

$$P_{ON\to OFF} = \rho\Big[1 - \frac{1}{1+e^{-\varepsilon(V_m - V_{OFF})}}\Big], \tag{3}$$

where, as previously mentioned, V_{ON} is the forward threshold voltage (when V_m exceeds this threshold, the memristor initiates the transition to LRS) and V_{OFF} is the reverse threshold

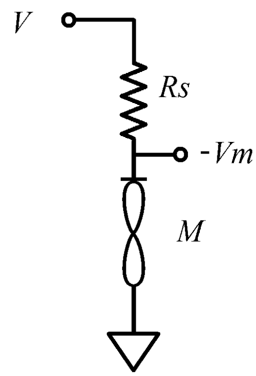

Fig. 3. Reference circuit for both simulations and measurements.

voltage (when V_m falls below this threshold the device initiates the transition to HRS); $\rho = dt/\tau$ is a fixed parameter, which is defined in terms of the memristor time constant τ; and, finally, ε is a parameter that depends on the temperature T, the Boltzmann constant k and the elementary charge constant q:

$$\varepsilon = \frac{q}{kT} \tag{4}$$

With all these elements in place, we can now define dX/dt as:

$$\frac{dX}{dt} = \frac{1}{\tau}\Big\{[\frac{1}{1+e^{-\varepsilon(V_m - V_{ON})}}](1-X) - \left[1 - \frac{1}{1+e^{-\varepsilon(V_m - V_{OFF})}}\right]X\Big\}, \tag{5}$$

so that the net resistance of the conductive channel, R, depends on the parallel flow through the switches ON and OFF, each with the characteristic resistance of its state, so that it is satisfied that

$$\frac{1}{R} = \frac{X}{R_{ON}} + \frac{1-X}{R_{OFF}} \tag{6}$$

The generalized version of this model, namely the Generalized Mean Metastable Switch (GMMS) model includes the effect of a parallel Schottky diode and its current, as a function of the voltage drop across the memristor. In this case, the resulting net current is expressed as:

$$I = \varphi I_m(V_m, t) + (1-\varphi) I_s(V_m), \tag{7}$$

where φ is a value between 0 and 1, so that there is no current contribution from the Schottky diode when φ=1. In turn, the Schottky diode current is the net result of the forward and reverse current components inside the diode, given by:

$$I_s(V_m) = A_f e^{B_f V_m} - A_r e^{B_r V_m}, \tag{8}$$

where A_f, A_r, B_f, and B_r are positive values that determine the exponential growth of both current components.

Since the GMMS model is the essential reference for the modeling and simulation of Knowm's SDC memristors, it is convenient to make a couple of remarks regarding its correct parameterization to obtain results in good agreement with reality. In general, when comparing the simulated hysteresis curve against the one observed experimentally, for example for the test circuit in Fig. 3, the GMMS model will result in a good approximation if the temperature value is adjusted with an offset of approximately -190 (in kelvin), at least with the W-SDC type memristors, which are the ones with which, in this work, the measurements were performed, a circumstance also reported in [23]. Fig. 4 exemplifies a typical simulation with the GMMS model.

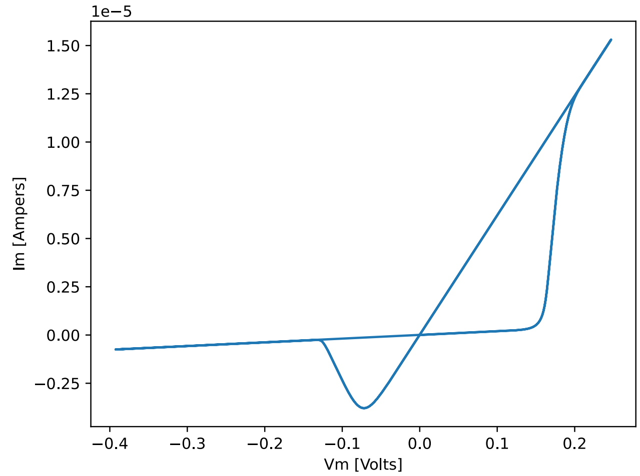

Fig. 4. Hysteresis curve obtained with the GMMS model, for the test circuit of Fig. 3, with $\tau = 6\times10^{-5}$s, T=(298.5-190)K, R_{ON}=13kΩ, R_{OFF}=460kΩ, R_s=10kΩ, V_{ON}=0.2V, V_{OFF}= -0.1V, $A_f = A_r = 10^{-7}$A, $B_f = B_r$=8 y φ=0.88, and a sinusoidal input signal V of 10Hz and 0.4V peak amplitude.

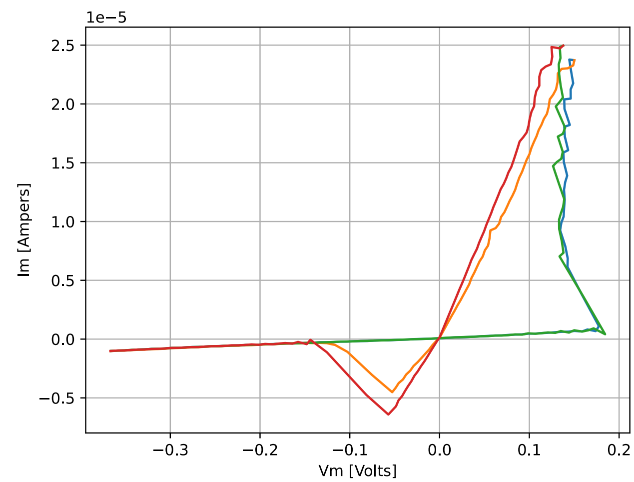

Fig. 5. Plots for two hysteresis cycles of a typical Knowm W-SDC memristor whose specifications are V_{ON}=0.26V (typical value) and V_{OFF}= -0.11V (typical value) and measured resistive values of R_{ON}=6kΩ, R_{OFF}=300kΩ. This experiment used the same test circuit and conditions corresponding to Fig. 4, but here the snapback phenomenon is evident.

Regarding the snapback phenomenon (see Fig. 5), it should be noted that it mostly occurs when the memristor is in optimal conditions (with HRS $>$ 400kΩ and LRS $<$ 4kΩ) and the device has not been put under too much operation stress, however, as the memristor degrades due to regular use,

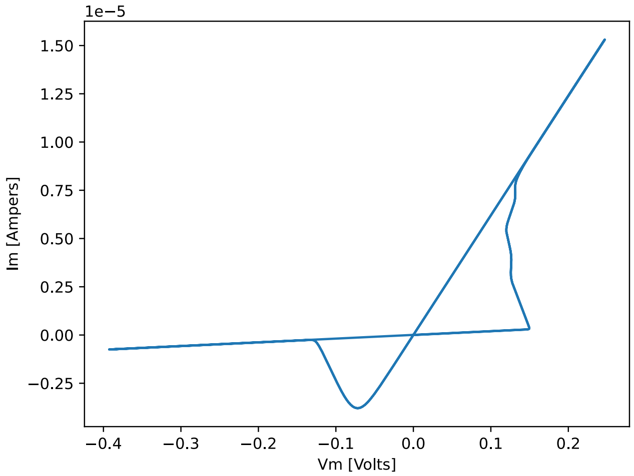

Fig. 6. Hysteresis curve obtained with the GMMS-Os model, for the test circuit of Fig. 3, with $\tau = 6 \times 10^{-5}$s, T=(298.5-190)K, R_{ON}=13kΩ, R_{OFF}=460kΩ, R_s=10kΩ, $V_{threshold}$=0.14V, V_{OFF}= -0.1V, $A_f = A_r = 10^{-7}$A, $B_f = B_r$=8 y φ=0.88, and sinusoidal input signal V of 10Hz and 0.4V peak amplitude.

the occurrence of this phenomenon becomes intermittent and may even disappear. To reproduce the snapback phenomenon, Ostrovskii et al. have made a proposal to adapt the original GMMS model [23]. Their proposal is to consider that the forward threshold voltage varies as a function of the state variable X according to the following expression:

$$V_{ON}(X) = V_{threshold} + \frac{0.1\cos(\frac{4\pi\sqrt{X}}{1.7-X})}{1+10\sqrt{X}}, \tag{9}$$

with $V_{threshold}$ the threshold level for V_{ON}. The GMMS model with the improvement of Ostrovskii et al., exclusively concerning the snapback phenomenon, is the one that will be used in this work as a reference model, and we will denote as GMMS-Os. Fig. 6 shows a typical simulation with this model (using the reference circuit shown in Fig. 3).

Regarding the phenomenon of threshold variability, both forward (V_{ON}) and reverse (V_{OFF}) voltages, we recall that Knowm has stated that this variability is a situation that derives from the eminently dynamic circumstances of the particular operation and history of each memristor [14], but no further details or explanations are provided. For the simulation of this variability Ostrovskii's team itself proposes to resort to a deterministic chaotic oscillator and various other proposals exist in the literature. Here we present an alternative approach based on an empirical characterization, under the hypothesis that the behavior of the variability of the thresholds can be well represented by using random variables of the α-stable type. Our objective is not only to reproduce the variability in the hysteresis cycles but, also to simulate this phenomenon in a practical and relatively simple way.

IV. Background Concepts on α-stable Random Variables

The Central Limit Theorem states that the normalized sum of a number of independent and identically distributed (i.i.d.)

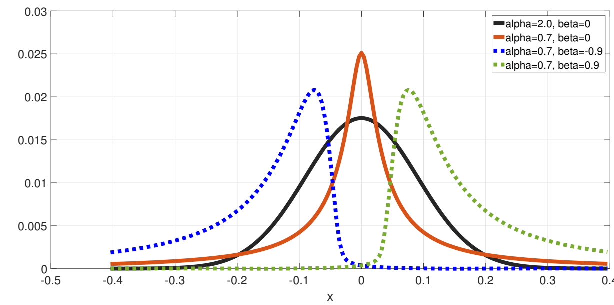

Fig. 7. Probability density functions (PDF) for α-stable random variables with different values for the parameters α and β. In all cases σ=1 and μ=0.

random variables with finite variance converges in distribution, as the number of terms increases, to a Gaussian distribution. A more general case arises when the condition related to the finite variance in removed from the theorem, leading to the Generalized Central Limit Theorem. In this case, the convergence also happens, but to a more general type of distributions, termed α-stable. As a consequence of this, the Gaussian random variable can be considered as a particular case of this family of distributions. Therefore, an α-stable random variable is suitable for representing, with appropriate parameterization, both normally distributed variables, as well as cases that include skewness or asymmetries and relatively long distribution tails (i.e., infinite variance) [31]. An α-stable variable is defined in terms of the stability parameter, α, such that $0 < \alpha \leqslant 2$; a skewness parameter, β, such that $-1 \leqslant \beta \leqslant 1$; a scale parameter, σ, with $\sigma \geqslant 0$ (not to be confused with the standard deviation); and a shift parameter, μ, which can be any real value. The stability parameter α can be viewed as an index indicating the rate at which the tails of the distribution decay. The α-stable random variable becomes a Gaussian random variable when α=2. For brevity, the notation $S_\alpha(\sigma, \beta, \mu)$ is used to refer to an α-stable distribution with the parameter values indicated by the subindex and in parentheses. As an example, Fig. 7 shows the profiles of some α-stable distributions.

Although there is no closed form expression for the probability density function (PDF) of an α-stable variable, it can be defined in terms of its characteristic function, namely [32]:

$$\Phi(\omega) = E[exp(j\omega x)] = \begin{cases} \exp\left[j\mu\omega - |\sigma\omega|^\alpha\left(1 - j\beta\frac{\omega}{|\omega|}\tan\left(\frac{\pi\alpha}{2}\right)\right)\right] & \text{if } \alpha \neq 1 \\ \exp\left[j\mu\omega - |\sigma\omega|\left(1 + j\beta\frac{2}{\pi}\frac{\omega}{|\omega|}\ln|\omega|\right)\right] & \text{if } \alpha = 1 \end{cases} \tag{10}$$

Equation (10) can be sampled by using different values of ω. Such discretized version of the characteristic function can be used to both estimate the α-stable parameters of a sequence of random values [31], [33] and to generate random sequences with an α-stable distribution [34]. The basic idea is to apply an inverse discrete Fourier transform, to obtain the sequence corresponding to the sampling of the Probability Density Function and, then, proceed with its numerical integration to obtain a sampled version of the corresponding Cumulative Distribution Function (CDF). The details, as to the practical realization of all this, can be found in the code `MonteCarlo_S_alpha_rand_core.py` that we provide and have made available to the interested reader [29].

V. Methodology for Estimation of Switching Variability Using α-stable Random Variables and Simulation

The methodology used to perform our work consisted of the following 4 steps (see Fig. 8), which can be replicated for steps 2 to 3 with the help of the Python code available in the repository [29]:

1) The *Analog Discovery 2* device as well as its SDK programming tools, both provided by the manufacturer Digilent, were used to perform automated measurements, on the circuit depicted in Fig. 3, for recording a number of hysteresis cycles of a Knowm W-SDC memristor. The purpose was to collect a sufficiently large number of voltage values for the observed V_{ON} and V_{OFF} thresholds to fit probability distributions.
2) From the data set for V_{ON} and V_{OFF}, the parameters of the α-stable random variables that achieve the best fit of the observed distributions were estimated. The `S_alpha_identification()` function, found in the `MonteCarlo_S_alpha_rand_core.py` code, was used for this purpose. For more details, see the example code `alpha-stable_demo.py`.
3) Assuming that the observed V_{ON} and V_{OFF} distributions can be represented by α-stable variables and given that these thresholds are parameters within the memristor model, they were incorporated within the simulation code, by means of stochastic processes (with the help of the functions within the `MonteCarlo_S_alpha_rand_core. py` and `MonteCarlo_Gaussian_rand_core.py`) as specified for the parameters α, β, σ and μ, of each of the corresponding α-stable random variables.
4) Finally, the simulation code was run, based on the model developed for the memristor, to corroborate that the behavior in the variations of the V_{ON} and V_{OFF} thresholds of the simulated hysteresis curves reproduces, in an acceptable way, the variability that was experimentally observed. For this, a code similar to the `stochastic_GMMS_memristor_demo.py` program and statistical tools were used to measure the error in the fit of the simulated distribution curves with respect to the real distributions.

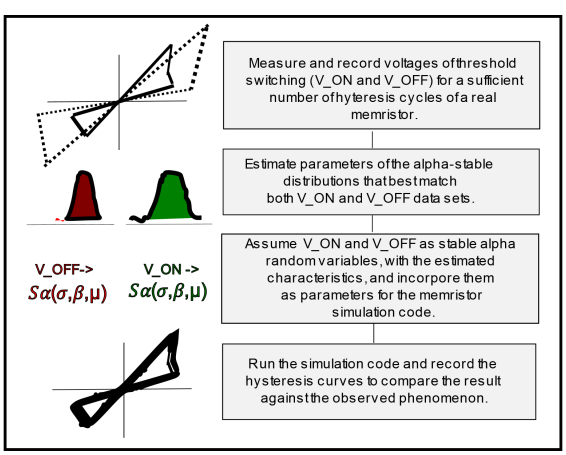

Fig. 8. Outline of the employed methodology.

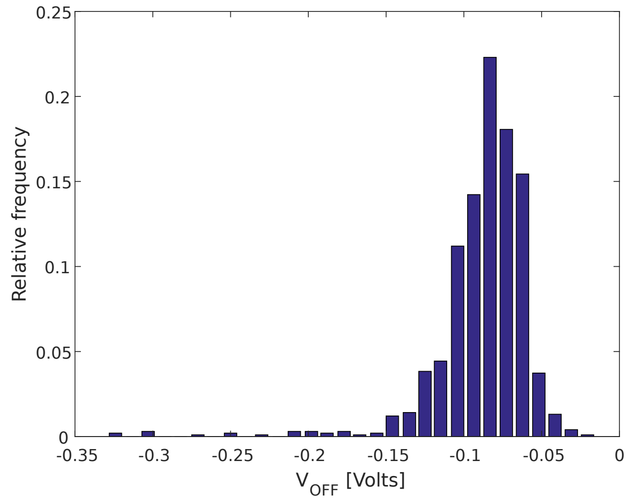

Fig. 9. Example of frequency distributions for reverse voltage threshold (V_{OFF}).

VI. Characterization of the Set and Reset Thresholds as α-stable Distributions

In order to exemplify the methodology to characterize the distributions with respect to the forward (V_{ON}) and reverse (V_{OFF}) threshold voltages, one of the W-SDC memristors, contained in a PCI-E-36 package, was employed. The initial resistance values were estimated to be around 300kΩ, for HRS, and around 6kΩ, for LRS. Hysteresis cycles (I_m vs. V_m) were recorded when the input signal was a triangular signal of 1Hz and varying in the interval ±0.4V. For this purpose, we used Knowm's Memristor Discovery 2.1 experimental platform [14] with the voltage divider configuration shown in Fig. 3. The plots of 1,000 hysteresis curves were obtained, based on the estimated current in the memristor (I_m), corresponding to the voltage drop recorded at the memristor ends (V_m), when the input voltage (V) was applied to the test circuit. For each hysteresis cycle, the threshold voltages were determined by inspection and by using the criterion of detecting significant and irreversible slope changes around the thresholds specified by the manufacturer. Fig. 9 shows the sample histogram of V_{OFF}, whereas Fig. 10 shows the corresponding one for V_{ON} (for visualization purposes only we used the method proposed by Scott [35], to estimate the number of bins used to generate each histogram, with a Gaussian density as a reference).

We proceeded to the α-stable characterization of the memristor thresholds based on the hypothesis that the probabilistic behavior can be well represented by a random variable of this type. We theorize that α-stable random variables are an appropriate way to represent the memristor on-off transitions in accordance with the GMMS model. This is due to the fact that, in the GMMS model, the memristor switching behavior is the result of the aggregate effect of a large number of metastable switches whose probabilistic behavior is controlled by the same pdf. As previously mentioned, the α-stable random variable is a limiting distribution arising from the superposition of a large number of i.i.d. random variables. Thus, there is a good agreement between the two conceptual frameworks. Table II shows an example of the results obtained from the parameter estimation procedures, which corresponds to one of the W-SDC memristors, available in the previously referred package.

Table II allow us to state that, in principle, threshold V_{OFF} behaves as an α-stable random variable of high variability (α= 1.62), whereas in the case of V_{ON}, the estimated value of α is 2.0 for all practical purposes, indicating that this is a random variable with a profile like that of a Gaussian variable. Note that, in this latter case, the value of β is irrelevant (see Eq. (10)) so that the parameter value resulting from the estimation procedure can be dismissed. We would like to remark

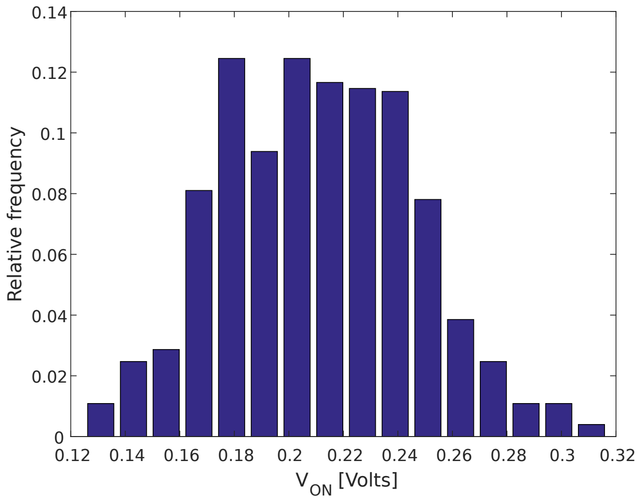

Fig. 10. Example of frequency distributions for forward voltage threshold (V_{ON}).

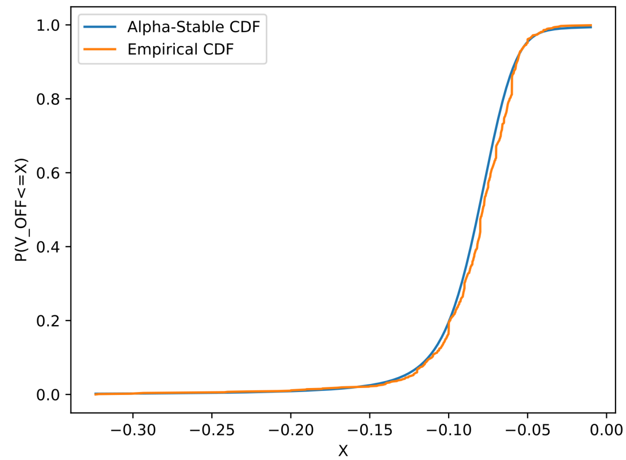

Fig. 11. Cumulative distribution functions for the V_{OFF} threshold, the α-stable model uses the following parameter values: α=1.62, β=-0.781, σ=0.0135V, and μ=-0.0826V.

TABLE II
ESTIMATION RESULTS FOR α-STABLE PARAMETERS

Threshold	α	β	σ [Volts]	μ [Volts]
V_{OFF}	1.62	-0.781	0.0135	-0.0826
V_{ON}	2.02	-1.503	0.0254	0.2169

that, in spite of the notation, the scale parameter σ of a Gaussian random variable is not equivalent to its standard deviation (let us denote the latter by σ_g). However, they are related since $\sigma_g = \sqrt{2}\sigma$ which yields 0.036V (this value can also be obtained by using the usual procedure for the computation of the sample standard deviation from the sample data). Also note that, in this case, the shift parameter (μ) corresponds to the mean (μ_g).

In Fig. 11, one can compare the empirical cumulative distribution (ECDF) curve for the V_{OFF} threshold with respect to the profile of the reference cumulative distribution function resulting from the estimated α-stable parameters. In turn, Fig. 12 shows the comparison between the ECDF obtained from the samples of V_{ON} and the CDF of a Gaussian model with its mean and standard deviation computed from the shift and scale parameters, respectively. By simple inspection of Figures 11 and 12, it can be observed that the α-stable characterizations of the on and off thresholds are in good agreement with the experimental data. In addition to these visual "proofs", some tests of goodness of fit were also carried out.

Regarding V_{OFF}, which exhibited high variability characterized by an index of stability $\alpha < 2$ (i.e., 1.62), a comment is in order. Heavy tails (as well as many other mathematical properties) are useful idealizations able to represent some characteristics that may be present in the sample data. However, whereas a heavy-tailed random variable can exhibit such behavior, real electric circuits have some limitations that should not be overlooked. For instance, voltage V_m in the voltage divider circuit of Fig. 3 cannot go beyond the applied voltage bias, even though an α-stable model may predict the occurrence of some sporadic samples of higher values. Nevertheless, within its applicability range, in our tests the α-stable model proved to be a useful approach to capture both significant deviations from the mean value and skewness.

For the V_{ON} threshold, the one-sample Kolmogorov-Smirnov (K-S) test was used to prove the hypothesis that a Gaussian model indeed fits the sample data. To this end, each sample of the trace was standardized by subtracting the sample mean and dividing by

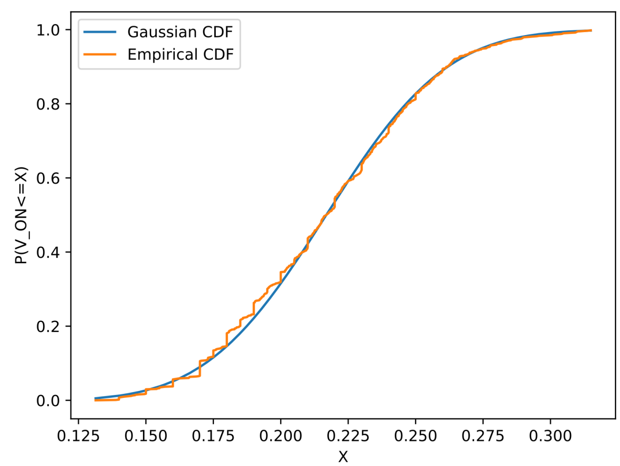

Fig. 12. Cumulative distribution functions for the V_{ON} threshold, the Gaussian model uses the following parameter values: μ_g=0.2169V and σ_g=0.036V.

the standard deviation. By means of the K-S test it was investigated whether an assumed standard Gaussian random variable (i.e., μ_g=0 and σ_g=1) fits the data or not. The test passed with a 5% probability (typical value) of incorrectly rejecting the hypothesis. In other words, with a 95% confidence level, the Gaussian random variable fits the sample data.

VII. PRACTICAL DEMONSTRATION OF A STOCHASTIC VERSION FOR GMMS-OS MODEL

In this section we demonstrate the practical utility of the procedure presented in this paper for simulation purposes. We will take advantage of the previously listed parameter estimates and will produce sequences of values with the distributions of interest incorporating these within the GMMS-Os model to introduce the desired variability in the V_{OFF} and V_{ON} thresholds. To this end, recall that a simple and efficient way to generate values with a certain probability distribution, with less computational time, although at the cost of requiring some reserved memory space, is to employ the Monte Carlo roulette

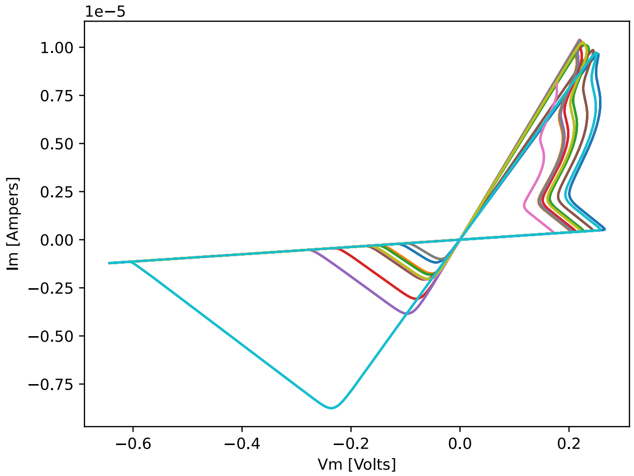

Fig. 13. Simulation results for 10 hysteresis cycles, for the test circuit of Fig. 3, employing the GMMS-Os model improved with a subjacent α-stable variable, with V_{OFF} and V_{ON} characterized according to the estimates in Table II and making $\tau = 6 \times 10^{-5}$s, T=(298.5-190)K, R_{ON}=13kΩ, R_{OFF}=460kΩ, R_s=10kΩ, $A_f = A_r = 10^{-7}$A, $B_f = B_r$=8 and φ=0.88, sinusoidal input signal V of 10Hz and 0.4V of peak amplitude.

principle, starting from tables of a suitable size, e.g. 1000 records, and an index that determines the retrieved value by means of a uniform distribution. In all cases, the table must store predetermined values, which correspond to an ordered sequence of samples and whose empirical distribution profile corresponds precisely to the desired random variable.

Fig. 13 shows the results obtained in the simulation of 10 hysteresis cycles, using an α-stable variable with α=1.62 to determine the value of V_{OFF}, and a Gaussian variable to determine the value of V_{ON}, both parameterized with the estimates shown in Table II. The procedure can be as simple as defining the variables representing V_{OFF} and V_{ON} as global variables and determining their value in each of the iterations of the simulation. Details of this practical realization can be found in the code (`stochastic_GMMS_memristor_demo.py`) used to produce the results shown in Fig. 13, which is available in the repository of [29].

The example depicted in Fig. 13 shows the asymmetry and the significant dispersion in values that can be obtained in the case of the threshold voltage V_{OFF} (this is the effect of using an α-stable variable of parameter α=1.62). In contrast, the threshold voltage V_{ON} shows a more regular and symmetric behavior due to the modeling using a Gaussian variable (i.e., α-stable with parameter α=2). The behavior obtained with the simulator is in complete agreement with our own observations and those made and reported in other works that have experimented with Knowm's W-SDC devices [10], [23], [26]. These results shown the enormous potential that the developed models have in reproducing, with higher fidelity, the behavior of memristor devices in simulation studies.

VIII. Conclusions

A practical methodology has been presented for the characterization of the set and reset threshold variability observed in Knowm's SDC memristors. To this end, large traces of the observed threshold voltages (i.e., V_{OFF} and V_{ON}) were recorded and, by assuming that their probabilistic behavior can be well represented by using α-stable distributions, the model parameters were estimated. This modelling approach turned out to be highly appropriate to capture the probabilistic behavior of the threshold voltages since this type of random variables are able to capture both skewness and high variability in the data, which were present in the sample traces. In addition, they also include the Gaussian distribution as a particular case. Experimental results demonstrate that the proposed methodology is useful and that, in the case of a typical Knowm's SDC memristors, both V_{OFF} and V_{ON} can be represented by α-stable random variables.

Additionally, as a practical reference, the GMMS model with the necessary random variables was coded to obtain a simple and computationally efficient stochastic model that reproduces, in an acceptable way, the variations observed in the threshold voltages of the memristors under study. The resulting model is intended to be used within the GMMS model. To allow interested researchers to experiment with our proposal, a demonstration code, available in a freely accessible repository, is provided for both the estimation of the α-stable parameters and the simulation with the stochastic variant of the GMMS-Os model.

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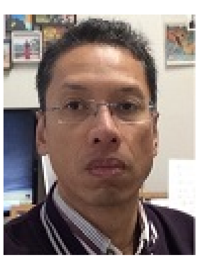

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