A Quasi-LPV Dynamic Output Feedback Stabilizer for Nonlinear Descriptor Systems via Convex Optimization Techniques

Arturo Alvarado, Tonatiuh Hernández-Cortés, Jaime González-Sierra, and Víctor Estrada-Manzo

***Abstract*—This work proposes stabilizing descriptor systems via an output feedback controller belonging to the dynamic category, that is, observer-based controllers. The proposed approach allows for handling nonlinear descriptor systems whose non-constant terms might depend on unavailable signals. The designing conditions are linear matrix inequalities obtained from applying convex representations in combination with the Lyapunov method. Numerical examples and real-time experiments in the well-known rotatory inverted pendulum illustrate the effectiveness of the methodology.**

Link to graphical and video abstracts, and to code:
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***Index Terms*—Quasi-LPV representations; Lyapunov method; nonlinear descriptor systems; output feedback; linear matrix inequality; rotatory inverted pendulum.**

I. Introduction

IMPLEMENTING state feedback controllers requires the availability of the whole state variables; nonetheless, this is not possible in several applications because the sensors are not available or have a high cost, among other reasons [1], [2]. A possible solution to this problem is using output feedback controllers, as this scheme employs the knowledge of the output (related to sensors) [3]. In the literature, one can find static and dynamic schemes [4]. Static output feedback controllers have been proven to be non-convex approaches [5], meaning that designing conditions cannot be directly cast as linear matrix inequalities (LMIs); this is an important shortcoming, as LMIs [6] are highly appreciated since they are solved in polynomial time [7]. Dynamic approaches may imply the design of a virtual system; in most cases, this system is a state observer [8], [9]. As the observer task is to provide the missing states used in the control law, the controller performance is achieved [10]. LMI-based observers have been widely studied from linear systems to nonlinear ones, for the latter case proving that the separation principle holds [11], [12], [13]. Other approaches within the adaptive fashion are the ones [14], [15].

On one hand, dealing with nonlinear or parameter varying systems, convex representations (Takagi-Sugeno (TS) [16], linear parameter varying (LPV) [17]) have been widely used as they, generally, lead to LMI conditions when adequately combined with the direct Lyapunov method [3], [7], [18], [19]. On the other hand, descriptor models have been shown to have a numerical advantage over traditional modeling [20], as in terms of vertex models (often called submodels) a lower number is required, especially with exact convex representation [21]; other advantages are the direct use of Lyapunov functions with extra decision variables [22], [23]. These characteristics of convex descriptor forms, in most cases, allow achieving feasible solutions whether other approaches cannot and/or better performance indexes. LMIs for linear descriptor and singular systems have been developed in [24], nonlinear descriptors have been considered in [25].

Contribution: A methodology for the design of a quasi-LPV dynamic output feedback controller for nonlinear descriptor system capable of driving the states to the origin with only available signals whether other approaches cannot; thus, in this cases fewer sensors can be used since some state variables can be estimated employing a quasi-LPV observer. Moreover, our approach provides relaxed results in comparison with former approaches [26], [27], thus allowing better performance indexes of the closed-loop system.

Organization: Section II presents the problem and some useful mathematical tools for solving it. Section III concerned with computation of an observer-based controller via LMIs and convex representations. Section IV illustrates the performance of our proposal concerning other approaches via some academic examples and real-time experiments in the well-known rotatory inverted pendulum system. Finally, some final comments are collected in Section V.

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A. Alvarado and Victor Estrada Manzo are with Department of Mechatronics of the Universidad Politécnica de Pachuca, C.P. 43830, Zempoala, Hidalgo, Mexico (e-mails: alvaradocampos@micorreo.upp.edu.mx, and victor_estrada@upp.edu.mx).

T. Hernández is with Universidad Autónoma del Estado de Hidalgo, Instituto de Ciencias Básicas e Ingeniería, Pachuca, Hidalgo, Mexico (e-mail: tonatiuh_hernandez@uaeh.edu.mx).

J. González is with the Unidad Profesional Interdisciplinaria de Ingeniería Campus Hidalgo, Instituto Politécnico Nacional, Carretera Pachuca—Actopan Kilómetro 1+500, Distrito de Educación, Salud, Ciencia, Tecnología e Innovación, C.P. 42162, San Agustín Tlaxiaca, Hidalgo, Mexico (e-mail: jagonzalezsi@ipn.mx).

II. Problem Statement and Preliminaries

Let us consider a nonlinear descriptor model[1]:

$$E(y)\dot{x}(t) = f(x) + g(x)u(t), \quad y(t) = h(x), \tag{1}$$

with the state vector denoted by $x \in \mathbb{R}^n$, the input vector is represented by $u \in \mathbb{R}^m$, and the output vector is $y \in \mathbb{R}^q$, $f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$, and $h(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^q$ are sufficiently smooth vector fields such that $f(0) = 0$, $g(0) = 0$, $h(0) = 0$, $E(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ is such that $\det(E(\cdot)) \neq 0$ for all $x \in \Omega_x \subset \mathbb{R}^n$, $0 \in \Omega_x$. This paper is concerned with the design of a controller with the form

$$u(t) = k(\hat{x}, y), \tag{2}$$

where $\hat{x} \in \mathbb{R}^n$ denotes a vector of estimated states. Note that the term $k(\cdot,\cdot)$ implies a nonlinear gain such that $k(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^q \mapsto \mathbb{R}^m$ is a function of available signals, i.e., it depends on the output y and some estimated states $\hat{x}$. Therefore, the control law (2) has to be designed such that the origin of

$$E(y)\dot{x}(t) = f(x) + g(x)k(\hat{x}, y), \tag{3}$$

is asymptotically stable.

The methodology requires designing a nonlinear observer

$$E(y)\dot{\hat{x}}(t) = f(\hat{x}) + g(\hat{x})u + L(\hat{x}, y)(y - \hat{y}), \quad \hat{y} = h(\hat{x}), \tag{4}$$

such that the estimation error $e = x - \hat{x}$ holds $\lim_{t\to\infty} e(t) = 0$, so $\hat{x}$ can be used in (2), where $L(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^q \mapsto \mathbb{R}^{n\times q}$ is the observer gain. Note that the controller and the observer gains are nonlinear; both designs will be cast as LMIs. As shown in [6], the separation principle could be applied, and therefore, both gains can be designed separately. To this end, the associated dynamic systems should be expressed in an exact convex/polytopic form.

A. Exact Polytopic Expressions

Nonlinear functions can be rewritten as polytopic/convex expressions of their bounds employing the sector nonlinearity [29]. Therefore, consider a region of interest Ω where non-constant terms $z(\cdot)$ are bounded, that is, $z \in [z^0, z^1]$; then, $z(\cdot) = w_0(z)z^0 + w_1(z)z^1$, where functions $w_0 = (z^1 - z(\cdot))/(z^1 - z^0)$ and $w_1 = 1 - w_0$ hold the convex sum property in the same region, i.e., $w_0(z) + w_1(z) = 1$ and $w_0, w_1 \in [0, 1]$. Thanks to convexity, this can be applied to vector/matrix expressions with many non-constant terms $z(\cdot)$, for instance, let $z_1(\cdot) = w_0^1(z)z_1^0 + w_1^1(z)z_1^1 = \sum_{i_1=0}^{1} w_{i_1}^1(z)z_1^{i_1}$, $z_2(\cdot) = w_0^2(z)z_2^0 + w_1^2(z)z_2^1 = \sum_{i_2=0}^{1} w_{i_2}^2(z)z_2^{i_2}$, therefore

$$\begin{bmatrix} z_1(\cdot) \\ z_1(\cdot) + z_2(\cdot) \end{bmatrix} = \begin{bmatrix} \sum_{i_1=0}^{1} w_{i_1}^1(z)z_1^{i_1} \\ \sum_{i_1=0}^{1} w_{i_1}^1(z)z_1^{i_1} + \sum_{i_2=0}^{1} w_{i_2}^2(z)z_2^{i_2} \end{bmatrix} = \sum_{i_1=0}^{1} w_{i_1}^1(z) \sum_{i_2=0}^{1} w_{i_2}^2(z) \begin{bmatrix} z_1^{i_1} \\ z_1^{i_1} + z_2^{i_2} \end{bmatrix}.$$

[1]A descriptor model could arise from Euler-Lagrange formulation $M(q)\ddot{q} + C_o(q, \dot{q})\dot{q} + G(q) = \tau$, where $q \in \mathbb{R}^{n_q}$ is the generalized coordinate, τ is the generalized force vector, $M(q)$ is the inertia matrix, $C_o(q, \dot{q})$ is the matrix of Coriolis, and $G(q)$ is the vector of gravitational forces [20], [28].

These ideas are used to algebraically express a nonlinear system as a convex combination of linear vertex models.

Notation. Time dependency will be omitted when it can be inferred from the context. For a given matrix Q, its transpose is denoted by $Q^\top$, $Q > 0$ means that Q is a positive definite matrix. Symmetric terms in matrix expressions are denoted by an asterisk, similarly to inline expressions, it stands for the symmetric terms on its left-hand side, e.g.,

$$\begin{bmatrix} A & B^T \\ B & D - E + D^T - E^T + F \end{bmatrix} = \begin{bmatrix} A & (*) \\ B & D - E + (*) + F \end{bmatrix}.$$

III. Main Results

As stated in the previous sections, the controller and observer gains are designed through convex expressions and the direct Lyapunov method, yielding LMI conditions. To this end, the descriptor model is put into an augmented system; this procedure is better known as *descriptor redundancy* approach [30], [22]. To do so, consider $\dot{x} = \dot{x}$ and $0 \times \ddot{x} = f(x) + g(x)u - E(y)\dot{x}$ or in matrix form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ f(x) + g(x)u(t) - E(y)\dot{x} \end{bmatrix}, \tag{5}$$

where I is the $n \times n$ identity matrix.

A. Controller Design

In this section, the design is carried out by considering that all the states are available; then, for implementation, the unavailable states will be replaced by their estimates.

Let us assume that $E(y)$, $f(x)$, and $g(x)$ can be rewritten in terms of $z(\cdot)$, that is, $E(y) = E(z)$, $f(x) = A(z)x$ and $g(x) = B(z)$ where $z(x) \in \mathbb{R}^s$ contains all the nonlinearities. There are infinite equivalent factorizations $f(x) = A(z)x$ leading to different premise vectors, but they should be chosen such that all their entries are well-defined in $\Omega_x \subset \mathbb{R}^n$, $0 \in \Omega_x$. Then, system (5) is rewritten as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A(z) & -E(z) \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B(z) \end{bmatrix} u. \tag{6}$$

Thus, as $x \in \Omega_x$, then each entry of the premise vector $z(x)$ is bounded, i.e., $z_i(x) \in [z_i^0,\ z_i^1]$, $i \in \{1, 2, \ldots, s\}$. Following the sector nonlinearity approach (see Subsection II-A), we have $z_i(x) = w_0^i(z)z_i^0 + w_1^i(z)z_i^1$, where

$$w_0^i(z) = (z_i^1 - z_i(x))/(z_i^1 - z_i^0), \ \ w_1^i(z) = 1 - w_0^i(z), \tag{7}$$

satisfies the convex sum property in Ω_x, i.e., $w_0^i(z) + w_1^i(z) = 1$, $w_0^i(z), w_1^i(z) \in [0, 1]$. Hence, the scheduling functions are

$$\mathbf{w}_i(z) = w_{i_1}^1(z_1)w_{i_2}^2(z_2)\cdots w_{i_s}^s(z_s), \tag{8}$$

with $i \in \{1, 2, \ldots, r\}$, $i_j \in \{0, 1\}$, $r = 2^s$; indexes $[i_1, i_2, \ldots, i_s]$ are selected such as they form an s-digit binary representation of $(i-1)$. These functions also hold the convex sum property in Ω_x, i.e., $\sum_{i=1}^{r} \mathbf{w}_i(z) = 1$ and $\mathbf{w}_i(z) \in [0, 1]$. Then, we have an exact quasi-LPV augmented system

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \sum_{i=1}^{r} \mathbf{w}_i(z) \left(\begin{bmatrix} 0 & I \\ A_i & -E_i \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B_i \end{bmatrix} u \right),$$

where $(A_i, B_i, E_i) = (A(z), B(z), E(z))\,|_{\mathbf{w}_i=1}$; using the notation $\Upsilon_{\mathbf{w}} = \sum_{i=1}^{r} \mathbf{w}_i(z)\Upsilon_i$ yields:

$$\bar{E}\dot{\bar{x}} = \bar{A}_{\mathbf{w}}\bar{x} + \bar{B}_{\mathbf{w}}u, \tag{9}$$

with $\bar{x}=\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $\bar{E}=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{\mathbf{w}}=\begin{bmatrix} 0 & I \\ A_{\mathbf{w}} & -E_{\mathbf{w}} \end{bmatrix}$, and $\bar{B}_{\mathbf{w}}=\begin{bmatrix} 0 \\ B_{\mathbf{w}} \end{bmatrix}$. Now, the nonlinear controller $u(t) = K(x)x(t)$ is also cast as in a convex form; it shares the same family of scheduling functions as the plant (8), this type of convex control law is known as Paralell-Distributed-Compensator (PDC) [31]:

$$u(t)=K(z)x(t)=\textstyle\sum_{j=1}^{r}\mathbf{w}_j(z)K_jx(t)=\bar{K}_{\mathbf{w}}\bar{x}(t), \tag{10}$$

where $K_j \in \mathbb{R}^{m\times n}$, $j \in \{1, 2, \ldots, r\}$ and $\bar{K}_{\mathbf{w}} = \begin{bmatrix} K_{\mathbf{w}} & 0 \end{bmatrix}$. Thus, we have the following closed-loop system

$$\bar{E}\dot{\bar{x}} = \left(\bar{A}_{\mathbf{w}} + \bar{B}_{\mathbf{w}}\bar{K}_{\mathbf{w}}\right)\bar{x}. \tag{11}$$

The following results establishes LMI conditions for designing (10).

Theorem 1. The origin $x = 0$ of the descriptor system (1), under the control law (10), is asymptotically stable if there exists $M_j \in \mathbb{R}^{m\times n}$ and $P_1 = P_1^T, P_{3j}, P_{4j} \in \mathbb{R}^{n\times n}$, $j \in \{1, 2, \ldots, r\}$ such that $P_1 > 0$ and

$$\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \quad \forall (i,j) \in \{1, 2, \ldots, r\}, \tag{12}$$

are feasible with

$$\Upsilon_{ij} = \begin{bmatrix} P_{3j}+P_{3j}^T & (*) \\ A_iP_1+B_iM_j-E_iP_{3j}+P_{4j}^T & -E_iP_{4j}-P_{4j}^TE_i^T \end{bmatrix}.$$

Then, the controller gains are as $K_j{=}M_jP_1^{-1}, j{\in}\{1, 2, \ldots, r\}$.

Proof. Consider a Lyapunov function candidate $V(x) = x^TP_1^{-1}x$, $P_1 \in \mathbb{R}^{n\times n}$, $P_1 > 0$; using the extended vector in (9), the previous function is expressed as

$$V = \bar{x}^T\bar{E}^T\bar{P}_{\mathbf{w}}^{-1}\bar{x}, \quad \bar{E}^T\bar{P}_{\mathbf{w}}^{-1} = \left(\bar{P}_{\mathbf{w}}^{-1}\right)^T\bar{E} \geq 0, \tag{13}$$

with $\bar{P}_{\mathbf{w}} = \begin{bmatrix} P_1 & 0 \\ P_{3\mathbf{w}} & P_{4\mathbf{w}} \end{bmatrix}, P_{3\mathbf{w}}, P_{4\mathbf{w}} \in \mathbb{R}^{n\times n}$, $\bar{P}_{\mathbf{w}}^{-1} = \left(\sum_{j=1}^{r}\mathbf{w}_j(z)\begin{bmatrix} P_1 & 0 \\ P_{3j} & P_{4j} \end{bmatrix}\right)^{-1}$. The time-derivative of (13) after replacing (11) gives

$$\dot{V} = \bar{x}^T\left(\left(\bar{A}_{\mathbf{w}}+\bar{B}_{\mathbf{w}}\bar{K}_{\mathbf{w}}\right)^T\bar{P}_{\mathbf{w}}^{-1} + \left(\bar{P}_{\mathbf{w}}^{-1}\right)^T\left(\bar{A}_{\mathbf{w}}+\bar{B}_{\mathbf{w}}\bar{K}_{\mathbf{w}}\right)\right)\bar{x},$$

applying congruence property with $\bar{P}_{\mathbf{w}}^T$ we have that $\dot{V} < 0$ if

$$\bar{P}_{\mathbf{w}}^T\left(\bar{A}_{\mathbf{w}} + \bar{B}_{\mathbf{w}}\bar{K}_{\mathbf{w}}\right)^T + \left(\bar{A}_{\mathbf{w}} + \bar{B}_{\mathbf{w}}\bar{K}_{\mathbf{w}}\right)\bar{P}_{\mathbf{w}} < 0,$$

holds. After the substitution of $\bar{M}_{\mathbf{w}} = \bar{K}_{\mathbf{w}}\bar{P}_{\mathbf{w}} = \begin{bmatrix} K_{\mathbf{w}}P_1 & 0 \end{bmatrix} = \begin{bmatrix} M_{\mathbf{w}} & 0 \end{bmatrix}$ gives

$$\textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}\mathbf{w}_i(z)\mathbf{w}_j(z)\left(\bar{A}_i\bar{P}_j + \bar{B}_i\bar{M}_j\right) + (*) < 0. \tag{14}$$

Finally, using the fact that the scheduling functions hold convex sum property in $\Omega_x \subset \mathbb{R}^n$, they can be dropped off using the well-known Tuan's relaxation Lemma[2]. □

[2]This option does not add slack matrices; thus, it avoids possible computational burden in comparison with other schemes [7, Appendix D].

B. Observer Design

Recall the augmented system (5) and consider the observer

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\hat{x}} \\ \dot{\chi} \end{bmatrix} = \begin{bmatrix} \chi \\ f(\hat{x})+g(\hat{x})u-E(y)\chi \end{bmatrix} + \begin{bmatrix} N_1(\hat{x},y) \\ N_2(\hat{x},y) \end{bmatrix}(y-\hat{y}), \tag{15}$$

where $\hat{y} = h(\hat{x})$, $\chi \in \mathbb{R}^n$ is defined such that $\chi \to \dot{\hat{x}}$ as t goes to infinity. Recall that the estimation error is $e(t) = x(t) - \hat{x}(t)$, its time derivative is $\dot{e}(t) = \dot{x}(t) - \dot{\chi}(t)$, thus by considering the plant (5) and observer (15), an augmented error system is computed by utilizing the methodology given in [32]:

$$\bar{E}\dot{\bar{e}} = \left(\bar{A}_e(\tilde{z},\tilde{\zeta}) - \bar{N}(\tilde{z})\bar{C}_e(\tilde{z},\tilde{\zeta})\right)\bar{e}, \tag{16}$$

where the augmented matrices are

$$\bar{e}=\begin{bmatrix} e \\ \dot{e} \end{bmatrix}, \bar{E}=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \bar{A}_e(\tilde{z},\tilde{\zeta})=\begin{bmatrix} 0 & I \\ A_e(\tilde{z},\tilde{\zeta}) & -E_e(\tilde{z}) \end{bmatrix},$$

$$\bar{N}(\tilde{z})=\begin{bmatrix} N_1(\tilde{z}) \\ N_2(\tilde{z}) \end{bmatrix}, \bar{C}_e(\tilde{z},\tilde{\zeta})=\begin{bmatrix} C_e^T(\tilde{z},\tilde{\zeta}) \\ 0 \end{bmatrix}^T, \text{ with}$$

$$A_e(\tilde{z},\tilde{\zeta})e = f(x)-f(\hat{x})+\left(g(x)-g(\hat{x})\right)u,$$

$$C_e(\tilde{z},\tilde{\zeta})e=h(x)-h(\hat{x}), \quad \text{and} \quad E_e(\tilde{z})\dot{e}=E(y)\dot{x}-E(y)\dot{\chi}.$$

As before, the premise vector $\tilde{z} \in \mathbb{R}^s$ gathers nonlinearities depending on only available signals ($\hat{x}$, y) while $\tilde{\zeta} \in \mathbb{R}^\sigma$ contains the rest of the signals, respectively. The previous factorization and the definition of the premise vector are not unique. However, in order to use the sector nonlinearity approach, it is required that the nonlinear terms in $A_e(\tilde{z},\tilde{\zeta})$ and $C_e(\tilde{z},\tilde{\zeta})$ are well-defined in a region containing the origin. Indeed, due to designing specifications, each entry of $\tilde{z}$ and $\tilde{\zeta}$ is bounded in $x \in \Omega_x \subset \mathbb{R}^n$ and $\hat{x} \in \Omega_{\hat{x}} \subset \mathbb{R}^n$, i.e., $\tilde{z}_i \in [\tilde{z}_i^0,\ \tilde{z}_i^1]$, $i \in \{1, 2, \ldots, s\}$, and $\tilde{\zeta}_k \in [\tilde{\zeta}_k^0,\ \tilde{\zeta}_k^1]$, $k \in \{1, 2, \ldots, \sigma\}$. Then, $\tilde{z}_i(\hat{x}, y)$, $i \in \{1, 2, \ldots, s\}$ and $\tilde{\zeta}_k(x, \hat{x}, y)$, $k \in \{1, 2, \ldots, \sigma\}$ can be rewritten as $\tilde{z}_i(\hat{x}, y) = \tilde{w}_0^i(\tilde{z})\tilde{z}_i^0 + \tilde{w}_1^i(\tilde{z})\tilde{z}_i^1$, $\tilde{\zeta}_k(x, \hat{x}, y) = \tilde{\omega}_0^k(\tilde{\zeta})\tilde{\zeta}_k^0 + \tilde{\omega}_1^k(\tilde{\zeta})\tilde{\zeta}_k^1$, where $\tilde{w}_0^i(\tilde{z}) = (\tilde{z}_i^1 - \tilde{z}_i(\hat{x}, y))/(\tilde{z}_i^1 - \tilde{z}_i^0)$, $\tilde{w}_1^i(\tilde{z}) = 1 - \tilde{w}_0^i(\tilde{z})$, $\tilde{\omega}_0^k(\tilde{\zeta}) = (\tilde{\zeta}_k^1 - \tilde{\zeta}_k(x, \hat{x}, y))/(\tilde{\zeta}_k^1 - \tilde{\zeta}_k^0)$, and $\tilde{\omega}_1^k(\tilde{\zeta}) = 1 - \tilde{\omega}_0^k(\tilde{\zeta})$ are functions holding the convex sum property in $\Omega_x \times \Omega_{\hat{x}}$, i.e., $\tilde{w}_0^i(\tilde{z}) + \tilde{w}_1^i(\tilde{z}) = 1$, $\tilde{w}_0^i(\tilde{z}), \tilde{w}_1^i(\tilde{z}) \in [0,\ 1]$, $\tilde{\omega}_0^k(\tilde{\zeta}) + \tilde{\omega}_1^k(\tilde{\zeta}) = 1$, $\tilde{\omega}_0^k(\tilde{\zeta})$, $\tilde{\omega}_1^k(\tilde{\zeta}) \in [0,\ 1]$. Hence, the so-called scheduling functions are defined as follows: $\tilde{\mathbf{w}}_i(\tilde{z}) = \tilde{w}_{i_1}^1(\tilde{z}_1)\tilde{w}_{i_2}^2(\tilde{z}_2)\cdots\tilde{w}_{i_s}^s(\tilde{z}_s)$, $\tilde{\boldsymbol{\omega}}_k(\tilde{\zeta}) = \tilde{\omega}_{k_1}^1(\tilde{\zeta}_1)\tilde{\omega}_{k_2}^2(\tilde{\zeta}_2)\cdots\tilde{\omega}_{k_\sigma}^\sigma(\tilde{\zeta}_\sigma)$, with $i \in \{1, 2, \ldots, r\}$, $k \in \{1, 2, \ldots, \rho\}$ $i_j \in \{0, 1\}$, $r = 2^s$, $\rho = 2^\sigma$; the sets of indexes $[i_1, i_2, \ldots, i_s]$ and $[k_1, k_2, \ldots, k_\sigma]$ are computed such that they are a binary representation of s-digit and σ-digit for $(i-1)$ and $(k-1)$; respectively. The scheduling functions are convex too. Finally, an exact quasi-LPV model of error system (16) is given as

$$\bar{E}\dot{\bar{e}} = \left(\bar{A}_{e\tilde{\mathbf{w}}\tilde{\boldsymbol{\omega}}} - \bar{N}(\tilde{z})\bar{C}_{e\tilde{\mathbf{w}}\tilde{\boldsymbol{\omega}}}\right)\bar{e}, \tag{17}$$

where the structure of the gain $\bar{N}(\tilde{z})$ will be defined latter and the short-hand notation $\Upsilon_{\tilde{\boldsymbol{\omega}}} = \sum_{k=1}^{\rho}\tilde{\boldsymbol{\omega}}_k(\tilde{\zeta})\Upsilon_k$ has been considered. Note that (16) and (17) are the same system, but the latter is a convenient convex form.

Remark 1. The nonlinearities involved in systems (5) and (16) are not necessarily the same. Thus, different convex representations with different scheduling vectors might result.

Now, we are ready to establish LMI conditions for the design of the nonlinear observer (4).

Theorem 2. The estimation error $e = x - \hat{x}$ between (1) and the observer (4), goes asymptotically to zero if there exist $P_{1e} = P_{1e}^T, P_{3ej}, P_{4ej} \in \mathbb{R}^{n\times n}$, $M_{1ej}, M_{2ej} \in \mathbb{R}^{n\times q}$, $j \in \{1, 2, \ldots, r\}$ such that $P_{1e} > 0$ and LMIs

$$\frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0, \quad \begin{array}{l}\forall (i,j) \in \{1,2,\ldots,r\}^2,\\ k \in \{1,2,\ldots,\rho\},\end{array} \tag{18}$$

are feasible with

$$\Upsilon_{ij}^k = \begin{bmatrix} P_{3ej}^T A_{eik} - M_{1ej} C_{eik} + (*) & (*) \\ P_{4ej}^T A_{eik} - M_{2ej} C_{eik} + P_{1e} - E_{ei}^T P_{3ej} & -P_{4ej}^T E_{ei} + (*) \end{bmatrix}.$$

If the conditions are feasible, the observer gain in (4) is:

$$L(\hat{x}, y) = \begin{bmatrix} E(y) & I \end{bmatrix} \bar{P}_{e\tilde{\mathbf{w}}}^{-T} \bar{M}_{e\tilde{\mathbf{w}}}, \tag{19}$$

with $M_{1ej}, M_{2ej} \in \mathbb{R}^{n\times q}$,

$$\bar{P}_{e\tilde{\mathbf{w}}}^{-T} = \left(\sum_{j=1}^{r} \tilde{\mathbf{w}}_j(\tilde{z}) \bar{P}_{ej}\right)^{-T} \quad \text{and} \quad \bar{M}_{e\tilde{\mathbf{w}}} = \sum_{j=1}^{r} \tilde{\mathbf{w}}_j(\tilde{z}) \begin{bmatrix} M_{1ej} \\ M_{2ej} \end{bmatrix}.$$

Proof. Consider a Lyapunov function candidate in terms of the augmented vector $\bar{e}$ that is

$$V(\bar{e}) = \bar{e}^T \bar{E}^T \bar{P}_{e\tilde{\mathbf{w}}} \bar{e}, \; \bar{P}_{e\tilde{\mathbf{w}}} = \begin{bmatrix} P_{1e} & 0 \\ P_{3e\tilde{\mathbf{w}}} & P_{4e\tilde{\mathbf{w}}} \end{bmatrix}, \; P_{1e} > 0, \tag{20}$$

and $\bar{E}^T \bar{P}_{e\tilde{\mathbf{w}}} = \bar{P}_{e\tilde{\mathbf{w}}}^T \bar{E} \geq 0$; since $V(e) = \bar{e}^T \bar{E}^T \bar{P}_{e\tilde{\mathbf{w}}} \bar{e} = e^T P_{1e} e$, it holds $V > 0$, $\forall e \neq 0$ is a valid Lyapunov function candidate. Taking the time-derivative of (20) along (17) yields

$$\dot{V} = \bar{e}^T \bar{P}_{e\tilde{\mathbf{w}}}^T \left(\bar{A}_{e\tilde{\mathbf{w}}\tilde{\boldsymbol{\omega}}} - \bar{N}(\tilde{z}) \bar{C}_{e\tilde{\mathbf{w}}\tilde{\boldsymbol{\omega}}}\right) \bar{e} + (*), \tag{21}$$

which after the substitution of $\bar{N}(\tilde{z}) = \bar{P}_{e\tilde{\mathbf{w}}}^{-T} \bar{M}_{e\tilde{\mathbf{w}}}$ gives

$$\dot{V} = \bar{e}^T \left(\bar{P}_{e\tilde{\mathbf{w}}}^T \bar{A}_{e\tilde{\mathbf{w}}\tilde{\boldsymbol{\omega}}} - \bar{M}_{e\tilde{\mathbf{w}}} \bar{C}_{e\tilde{\mathbf{w}}\tilde{\boldsymbol{\omega}}}\right) \bar{e} + (*). \tag{22}$$

Now, $\dot{V} < 0$ is guaranteed if the following holds

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{\rho} \tilde{\mathbf{w}}_i(\tilde{z}) \tilde{\mathbf{w}}_j(\tilde{z}) \tilde{\boldsymbol{\omega}}_k(\tilde{\zeta}) \left(\bar{P}_{ej}^T \bar{A}_{eik} - \bar{M}_{ej} \bar{C}_{eik}\right) + (*) < 0$$

holds. Finally, once the scheduling functions are removed via Tuan's relaxation Lemma, the LMI conditions (18) yield and thus concluding the proof. □

Theorems 1 and 2 provide LMI conditions to construct the controller (2); nonetheless, in some cases, adding performance indexes such as decay rate, bounds on inputs and/or outputs, among others, are necessary to achieve improvements in closed-loop performance. In that sense, the next section provides some comments on the matter.

C. Implementation

The observer-based control law (2) is nonlinear but has a convex structure (convex scheduling functions depending on available and estimated signals, vertex gain matrices). Its design is carried out by conditions in Theorems 1 and 2; both sets of conditions can be checked simultaneously. Results from Theorem 1 provide the vertex gains $K_j \in \mathbb{R}^{m\times n}$, $j \in \{1, 2, \ldots, r\}$. Conditions in Theorem 2 allow constructing a state observer that estimates the missing state variables $\hat{x}$. Thus, the quasi-LPV dynamic output feedback control law to be implemented is

$$u = k(\hat{x}, y) = K(x_a)\hat{x} = \sum_{j=1}^{r} \mathbf{w}_j(z(x_a)) K_j \hat{x}, \tag{23}$$

where $x_a(t)$ is a state vector with only available signals, their estimates replace the missing states. For example, if $x \in \mathbb{R}^3$, with $y = x_2$, then $x_a = [\hat{x}_1 \;\; x_2 \;\; \hat{x}_3]^T$. Another important point for real-time experiments is the number of gains in (23), i.e., conditions in Theorem 1 have been derived by considering the maximun number of vertex gains (a traditional PDC). Nonetheless, the number of vertex gains K_j can be reduced even up to 1 (linear controller $u = K_1\hat{x}$), this reduces the computational burden but it may also reduce the feasibility of the corresponding LMI problem [15].

In general, the separation principle does not hold for nonlinear systems, but as proven in [6, Section 7.6] and [7, Theorem 6.11] in some instances it is possible, for example the approach in [33]; due to the controller and observer structure developed, our approach belongs to those holding the separation principle. Fig. 1 illustrates the proposed dynamic output feedback controller scheme (23).

Remark 2. In real-time applications, the convergence rate of the states and the error signal toward the origin can be guaranteed within the same LMI framework [7]. This can be done by checking $\dot{V}(x) + 2\beta_c V(x) < 0$, $\beta_c > 0$ and $\dot{V}(e) + 2\beta_o V(e) < 0$, $\beta_o > 0$ leading to conditions (12) with

$$\Upsilon_{ij} = \begin{bmatrix} P_{3j} + P_{3j}^T + 2\beta_c P_1 & (*) \\ A_i P_1 + B_i M_j - E_i P_{3j} + P_{4j}^T & -E_i P_{4j} - P_{4j}^T E_i^T \end{bmatrix}$$

and (18) with

$$\Upsilon_{ij}^k = \begin{bmatrix} P_{3ej}^T A_{eik} - M_{1ej} C_{eik} + (*) + 2\beta_o P_{1e} & (*) \\ P_{4ej}^T A_{eik} - M_{2ej} C_{eik} + P_{1e} - E_{ei}^T P_{3ej} & -P_{4ej}^T E_{ei} + (*) \end{bmatrix};$$

respectively. They can be seen as an LMI region problem [34].

Remark 3. Increasing the speed converge requires more energy from the actuators; input bounds $|u(t)| \leq \mu$, $\mu > 0$ can be imposed via LMIs, that is,

$$\begin{bmatrix} 1 & (*) \\ x(0) & P_1 \end{bmatrix} \geq 0 \quad \text{and} \quad \begin{bmatrix} P_1 & (*) \\ M_j & \mu^2 I \end{bmatrix} \geq 0 \tag{24}$$

where $x(0)$ is the vector of initial conditions [7]. A similar procedure can be also applied to bound the observer gain magnitude.

Conditions in Remarks 2 and 3 can be directly added to the LMI problems in Theorems 1 and 2 and run together as a single LMI problem [35].

IV. Examples

This section illustrates the proposal applicability; first examples are taken from the literature, while the last one considers real-time experiments in the well-known rotatory inverted pendulum. All computations have been performed in MATLAB r2015a and the LMIToolbox [36].

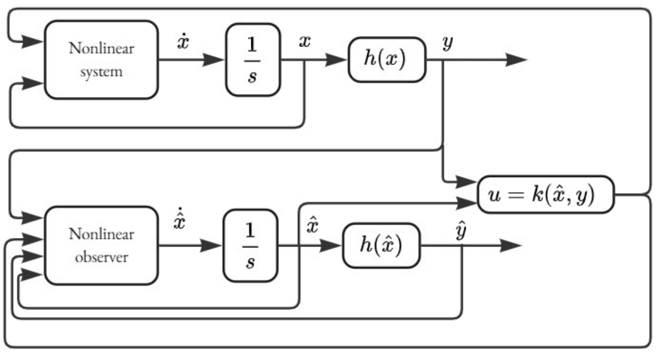

Fig. 1. Block diagram of the quasi-LPV dynamic output feedback controller (23), it is fed with the output $y(t)$ of the nonlinear system and $\hat{x}(t)$ from the nonlinear observer.

Example 1. Let us consider a continuous-time nonlinear system given in [26, Section IV], i.e., (1), with $E(y) = I$ and

$$f(x)=\begin{bmatrix} x_1+x_2+\sin x_3-0.1x_4 \\ x_1-2x_2 \\ x_1+x_1^2x_2-0.3x_3 \\ \sin x_3-x_4 \end{bmatrix}, \quad g(x)=\begin{bmatrix} x_1^2+1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and $h(x) = [x_2 + (x_1^2+1)x_4 \;\; x_1]^T$; thus the outputs are $y_1 = x_2 + (x_1^2+1)x_4$ and $y_2 = x_1$, note that the only state available from measurements is $x_1(t)$. In order to compare with [26], let us consider the region $\Omega_x = \{x \in \mathbb{R}^4 : |x_1| \leq 1.4, |x_3| \leq 0.7\}$. Following the methodology in Section III-A two different nonlinear terms can be identified in matrices $A(x)$, $B(x)$, and $C(x)$, i.e., $z_1 = x_1^2 \in [0 \; 1.96]$ and $z_2 = \sin x_3/x_3 \in [0.9203 \; 1]$. The vertex are

$$A_1=\begin{bmatrix} 1 & 1 & z_2^0 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & z_1^0 & -0.3 & 0 \\ 0 & 0 & z_2^0 & -1 \end{bmatrix}, A_2=\begin{bmatrix} 1 & 1 & z_2^1 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & z_1^0 & -0.3 & 0 \\ 0 & 0 & z_2^1 & -1 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 1 & 1 & z_2^0 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & z_1^1 & -0.3 & 0 \\ 0 & 0 & z_2^0 & -1 \end{bmatrix}, A_4=\begin{bmatrix} 1 & 1 & z_2^1 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & z_1^1 & -0.3 & 0 \\ 0 & 0 & z_2^1 & -1 \end{bmatrix},$$

$$B_1=B_2=\begin{bmatrix} z_1^0+1 & 0 & 0 & 0 \end{bmatrix}^T, B_3=B_4=\begin{bmatrix} z_1^1+1 & 0 & 0 & 0 \end{bmatrix}^T,$$

where the scheduling functions are $\mathbf{w}_1 = w_0^1 w_0^2$, $\mathbf{w}_2 = w_0^1 w_1^2$, $\mathbf{w}_3 = w_1^1 w_0^2$, and $\mathbf{w}_4 = w_1^1 w_1^2$, the weighting functions as in (7). Conditions in Theorem 1 together with performance indexes such as decay rate $\beta_c = 0.5$ and constraint $|u(t)| \leq 5$ for $x(0) = [-1.2 \; 0.5 \; 0.7 \; -0.6]^T$ were found feasible with:

$$P_1=\begin{bmatrix} 4.17 & 0.23 & -4.03 & -2.88 \\ 0.23 & 3.67 & -1.07 & -2.04 \\ -4.03 & -1.07 & 7.65 & 4.51 \\ -2.88 & -2.04 & 4.51 & 18.35 \end{bmatrix}, K_1^T=\begin{bmatrix} -2.87 \\ -1.28 \\ -1.83 \\ 0.08 \end{bmatrix},$$

$$K_2^T=\begin{bmatrix} -2.85 \\ -1.27 \\ -1.86 \\ 0.08 \end{bmatrix}, K_3^T=\begin{bmatrix} -1.58 \\ -0.87 \\ -1.12 \\ 0.05 \end{bmatrix}, K_4^T=\begin{bmatrix} -1.57 \\ -0.87 \\ -1.14 \\ 0.06 \end{bmatrix}.$$

As for the observer we have that $g(y)u - g(y)u = 0$,

$$f(x)-f(\hat{x})=A_e(\tilde{z},\tilde{\zeta})e=\begin{bmatrix} 1 & 1 & 1-1/6\tilde{\zeta}_1 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & \tilde{z}_1 & -0.3 & 0 \\ 0 & 0 & 1-1/6\tilde{\zeta}_1 & -1 \end{bmatrix}e$$

$$\text{and} \quad h(x)-h(\hat{x}) = C_e(\tilde{z},\tilde{\zeta})e = \begin{bmatrix} 0 & 1 & 0 & \tilde{z}_1+1 \\ 1 & 0 & 0 & 0 \end{bmatrix}e;$$

respectively, where $\sin x_3 - \sin \hat{x}_3 \approx (1 - 1/6(x_3^2 + x_3\hat{x}_3 + \hat{x}_3^2))e_3$. The premise variables are defined as $\tilde{z}_1 = x_1^2$ and $\tilde{\zeta}_1 = x_3^2 + x_3\hat{x}_3 + \hat{x}_3^2$ within Ω_x and $\Omega_{\hat{x}} = \{\hat{x} \in \mathbb{R}^4 : |\hat{x}_1| \leq 1.4, |\hat{x}_3| \leq 0.7\}$; then $r = 2$ and $\rho = 2$. Conditions in Theorem 2 have been run with $\beta_o = 1$ (see Remark 2); they yielded

$$P_{1e}=\begin{bmatrix} 22.73 & 1.22 & -6.40 & 2.44 \\ 1.22 & 18.49 & -0.97 & 0.23 \\ -6.40 & -0.97 & 4.63 & -3.35 \\ 2.44 & 0.23 & -3.35 & 7.67 \end{bmatrix},$$

$$M_{1e1}=\begin{bmatrix} -431.13 & -270.47 & 98.13 & -70.31 \\ -320.79 & -549.97 & 34.27 & -211.78 \end{bmatrix}^T,$$

$$M_{1e2}=\begin{bmatrix} -431.13 & -917.06 & 96.94 & -71.41 \\ -873.02 & 59.49 & 34.90 & -55.34 \end{bmatrix}^T,$$

$$M_{2e1}=\begin{bmatrix} 85.56 & -91.26 & 329.43 & -82.20 \\ 379.09 & -158.49 & 149.26 & -355.81 \end{bmatrix}^T,$$

$$M_{2e2}=\begin{bmatrix} 84.97 & -92.08 & 329.86 & -81.24 \\ 443.63 & 328.95 & 150.18 & -419.63 \end{bmatrix}^T.$$

Fig. 2 shows the error for initial conditions $x(0) = [-1.2 \;\; 0.5 \;\; 0.7 \;\; -0.6]^T$ and $\hat{x}(0) = 0$ while Fig. 3 plots the closed-loop trajectories, it can be seen that they converge asymptotically to the origin. Note that the nonlinear observer (4) feeds the controller (23) with $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$, that is, $x_a = [x_1 \;\; \hat{x}_2 \;\; \hat{x}_3 \;\; \hat{x}_4]^T$ (see the scheme given in Fig. 1). Let us compare the controller performance in terms of the control effort; recall that the controller reported in [26] is a static output feedback one. Fig. 4 shows the input signals for both cases, it can be seen that our proposal provides a smother control signal with a lower magnitude (holding the imposed constraint $\mu = 5$, see Remark 3). Moreover, conditions in [26] are not convex since a set of linear equations must be solved together with LMIs.

Example 2. Consider the example in [27, Example 7], the system therein has only one nonlinearity that depends on measurable signals, which in turn implies that associated nonlinear system (1) has the form $E(y)\dot{x} = A(y)x + g(y)u$, i.e., $f(x) = A(y)x$. Therefore, the convex descriptor models

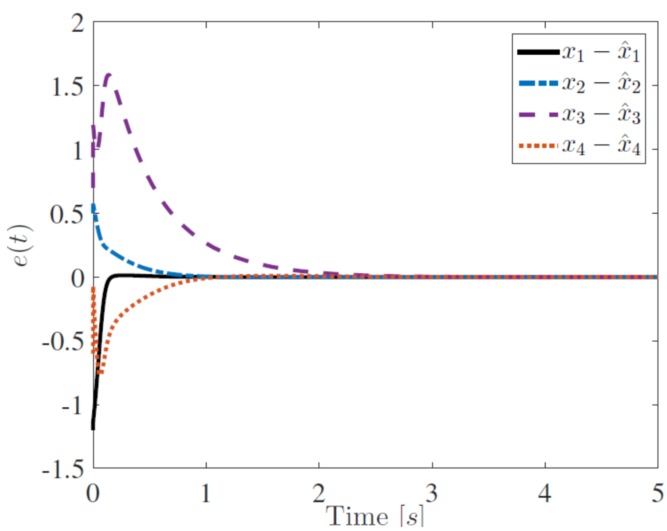

Fig. 2. Time evolution of the error signal $e(t) = x(t) - \hat{x}(t)$ in Example 1.

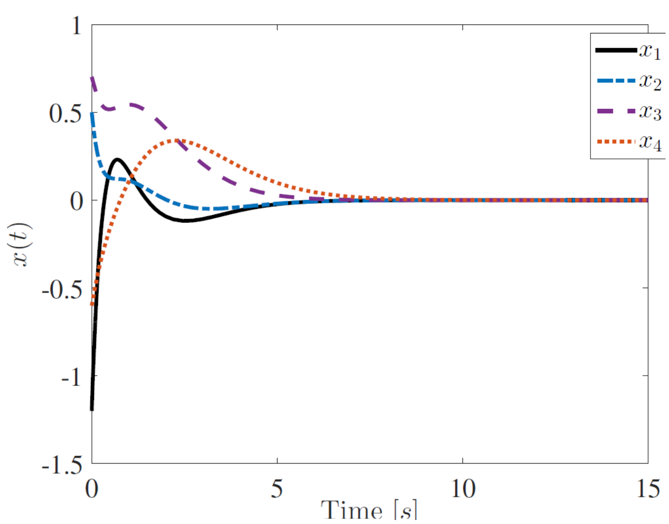

Fig. 3. Time evolution of the states $x(t)$ in Example 1.

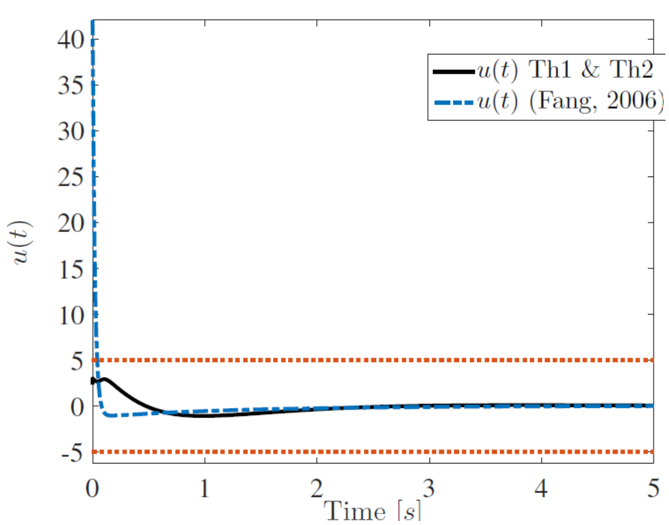

Fig. 4. Control signals from different approaches in Example 1.

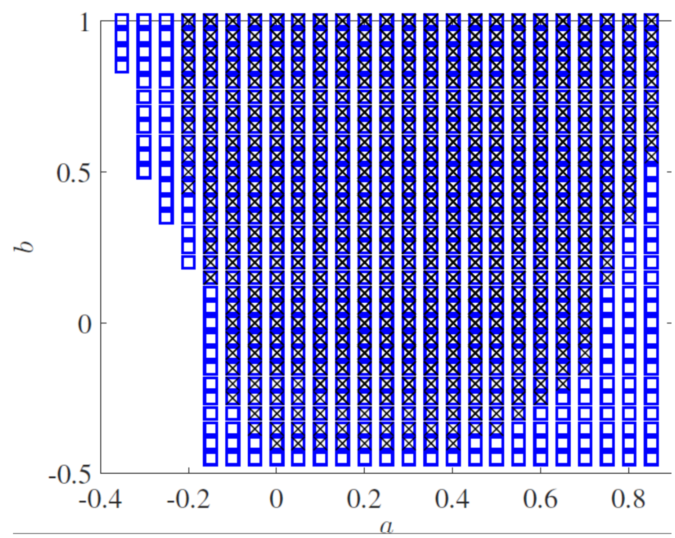

Fig. 5. Feasible set solution in Example 2.

(9) and (17) have vertex matrices

$$A_1{=}A_{e1}{=}\begin{bmatrix}-1.15 & 0.1 & 1.8+b\\ 0.3 & -1.3 & -0.5\\ -0.1 & 0.8 & -0.8\end{bmatrix}, B_1{=}\begin{bmatrix}0.6 & 1.2\\ 0.3 & 1.5-a\\ -0.6 & 1.3\end{bmatrix},$$

$$A_2{=}A_{e2}{=}\begin{bmatrix}-1.2 & -0.3 & -0.1\\ 0.4 & -0.6 & 0.3\\ -0.2 & -0.2 & -0.2-a\end{bmatrix}, B_2{=}\begin{bmatrix}-1.3 & 2.1\\ -2.7 & 0.5\\ 1.5 & 1.6\end{bmatrix},$$

$$E_1{=}E_{e1}{=}\begin{bmatrix}1.05 & 0.7 & 0.7\\ -0.1 & 1.1 & -0.2\\ 0.1 & 0.5 & 0.9{-}a\end{bmatrix}, C_1^T = C_{e1}^T{=}\begin{bmatrix}0.4\\ 1\\ 0\end{bmatrix},$$

$$E_2{=}E_{e2}{=}\begin{bmatrix}0.9{+}b & 0.8 & 0.77\\ -0.9 & 1.1 & -0.2\\ 0.4 & 0.5 & 0.6\end{bmatrix}, \text{ and } C_2^T = C_{e2}^T{=}\begin{bmatrix}0.8\\ 1\\ 0\end{bmatrix};$$

the parameters vary as $a \in [-0.5,\ \ 1]$ and $b \in [-0.5,\ \ 1]$. Note that, for the observer design (Theorem 2), we have $r = 2$ and $\rho = 0$; hence $A_{e\mathbf{w}\boldsymbol{\omega}} = A_{\mathbf{w}}$, $E_{e\mathbf{w}\boldsymbol{\omega}} = E_{\mathbf{w}}$, and $C_{e\mathbf{w}\boldsymbol{\omega}} = C_{\mathbf{w}}$. Fig. 5 compares the feasible solution sets for two approaches; clearly our proposal (blue-$\square$) produces a larger solution set than the static one (black-$\times$), i.e., using our conditions allows designing a controller for systems that cannot be stabilized with [27].

A. Real-time experiments: the rotatory inverted pendulum

Example 3. Consider the model of the Quanser Innovate Educate rotary inverted pendulum (RIP), whose Euler-Lagrange model is $M(q)\ddot{q} + C_o(q,\dot{q})\dot{q} + G(q) = \tau$, where $q = \begin{bmatrix}\theta & \alpha\end{bmatrix}^\top \in \mathbb{R}^2$ with θ as the angle of the arm measured from the $X-$axis and α the pendulum angle measured from a parallel line to the $Z-$axis (see Fig. 6); matrices are

$$M(q){=}\begin{bmatrix}T_1\!\left(0.25T_2\left(1{-}\cos^2 q_2\right){+}J_r{+}T_3\right) & -0.5T_1T_4\cos q_2\\ -0.5T_4\cos q_2 & T_6\end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix}0.5T_1T_2\dot{q}_2\sin q_2\cos q_2{+}T_5 & 0.5T_1T_4\dot{q}_2\sin q_2\\ -0.25T_2\dot{q}_1\cos q_2\sin q_2 & B_p\end{bmatrix},$$

$$G(q){=}\begin{bmatrix}0 & -0.5m_pL_pg\sin q_2\end{bmatrix}^T,$$

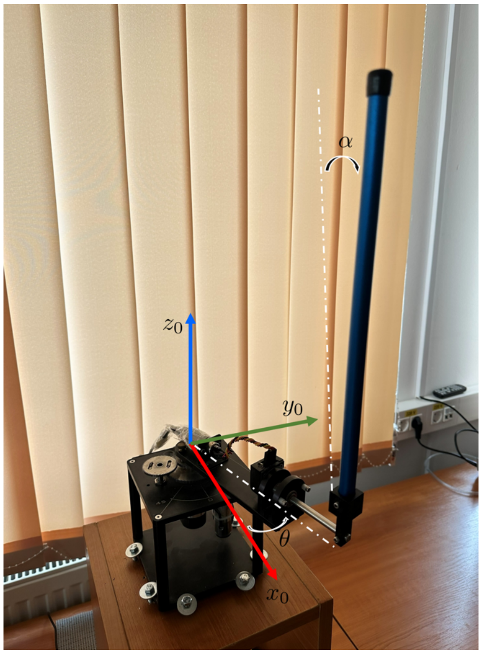

Fig. 6. The rotatory inverted pendulum system.

where $T_1 = R_m/(\eta_g k_g \eta_m k_t)$, $T_2 = m_p L_p^2$, $T_3 = m_p L_r^2$, $T_4 = m_p L_p L_r$, $T_5 = K_g k_m + T_1 B_r$, $T_6 = J_p + \frac{1}{4}T_2$, the parameter values are shown in Table I. Note that, the system input is the voltage applied to the DC motor, the generalized force vector is $\tau = \begin{bmatrix} V_m & 0 \end{bmatrix}^T$.

TABLE I
PARAMETER VALUES OF THE ROTATORY INVERTED PENDULUM

Symbol	Name	Value
g	gravitational acceleration	9.81 m/s^2
k_t	motor torque constant	0.00767 N·m
k_m	back EMF constant	0.00767 V/(rad/s)
K_g	total gear ratio	70
R_m	armature resistance	2.6 Ω
η_g	gear efficiency	0.9
η_m	motor efficiency	0.69
L_p	pendulum true length	0.3365 m
m_p	pendulum mass	0.127 kg
J_p	pendulum inertia	0.0012 kg·m^2
B_p	pendulum damping coefficient	0.0024 $N\cdot m\cdot s/rad$
L_r	rotary arm true length	0.2159 m
m_r	rotary arm mass	0.257 kg
J_r	rotary arm inertia	9.9829e^{-4} kg·m^2
B_r	rotary arm damping coefficient	0.0024 $N\cdot m\cdot s/rad$

From Euler-Lagrange equations a nonlinear descriptor can be computed by choosing $x_1 = q_1$, $x_2 = q_2$, $x_3 = \dot{q}_1$, and $x_4 = \dot{q}_2$ yielding (1) with matrices

$$E(y)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \begin{matrix} 0.25T_1T_2\left(1-\cos^2 x_2\right) \\ +T_1J_r+T_1T_3 \end{matrix} & -0.5T_1T_4\cos x_2 \\ 0 & 0 & -0.5T_4\cos x_2 & T_6 \end{bmatrix},$$

$$f(x)=\begin{bmatrix} x_3 \\ x_4 \\ -0.5T_1\sin x_2\left(T_2x_3x_4\cos x_2 + T_4x_4^2\right) - T_5x_3 \\ 0.5m_pL_pg\sin x_2 + 0.25T_2x_3^2\cos x_2\sin x_2 - B_px_4 \end{bmatrix},$$

$g(x) = [0\ 0\ 1\ 0]^T$, the output is $y = [x_1 \quad x_2]^T$.

In order to apply the methodology for the controller design in Section III-A a rewriting must be done $f(x) = A(z)x$ where the scheduling vector is $z_1 = \sin x_2$, $z_2 = \cos x_2$, $z_3 = \sin x_2/x_2$, $z_4 = x_3$, $z_5 = x_4$, $z_6 = \cos^2 x_2$, and the corresponding $2^6 = 64$ vertex matrices; nevertheless, in real-time implementations this might lead to computational burden. Following the recommendations in Section III-C, a controller (23) with only 8 vertex will be designed, that is, $u(t) = \sum_{j=1}^{8} \mathbf{w}_j(z(x_a)K_j\hat{x}$, where $z(x_a) = [\sin x_2,\ \cos x_2,\ \sin x_2/x_2]^T$, the rest of the premise variables are considered in the design process but not in the control law. Now, let us define an operating region

$$\Omega_x = \{x \in \mathbb{R}^4 : |x_2| \le 14^o, |x_3| \le 5\text{rad/s}, |x_4| \le 10\text{rad/s}\},$$

where the 64 vertex models are calculated. Note that the input matrix is constant $g(x) = B$; thus, choosing $\bar{P}_{\mathbf{w}} = \bar{P}$ for the Lyapunov function candidate (13) there are no double convex sums in (14). Increasing the speed converge towards the upright position of the vertical beam, a decay rate $\beta_c = 0.4$ has been set as well as an input bound $\mu = 10$ (see Remarks 2 and 3), it yields the following:

$$P_1 = 1\times 10^3\begin{bmatrix} 0.04 & -0.01 & -0.07 & 0.07 \\ -0.01 & 0.03 & 0.05 & -0.22 \\ -0.07 & 0.05 & 0.57 & -0.08 \\ 0.07 & -0.22 & -0.08 & 1.66 \end{bmatrix},$$
$$K_1=\begin{bmatrix} 0.81 & -17.81 & 1.53 & -2.37 \end{bmatrix},$$
$$K_3=\begin{bmatrix} 0.81 & -17.76 & 1.54 & -2.36 \end{bmatrix},$$
$$K_6=\begin{bmatrix} 0.80 & -17.77 & 1.53 & -2.36 \end{bmatrix},$$
$$K_8=\begin{bmatrix} 0.80 & -17.73 & 1.54 & -2.35 \end{bmatrix}.$$

With respect to the observer design (see Section III-B), we have that $f(x) - f(\hat{x}) = A_e(\tilde{z}, \tilde{\zeta})e$ with

$$A_e(\cdot,\cdot)=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.5T_1T_2\tilde{z}_1\tilde{z}_2\tilde{z}_4 & -0.5T_1T_2\tilde{z}_1(\tilde{z}_2\tilde{\zeta}_1+\tilde{\zeta}_2+\tilde{z}_4) \\ 0 & 0 & 0.25T_2\tilde{z}_1\tilde{z}_2(\tilde{\zeta}_1+\tilde{z}_3) & -B_p \end{bmatrix},$$

where $\tilde{z}_1 = \sin x_2$, $\tilde{z}_2 = \cos x_2$, $\tilde{z}_3 = \hat{x}_3$, $\tilde{z}_4 = \hat{x}_4$, $\tilde{z}_5 = \cos^2 x_2$, $\tilde{\zeta}_1 = x_3$ and $\tilde{\zeta}_2 = x_4$ have been chosen as scheduling vectors, their bounds have been computed within $\Omega_{\hat{x}} = \{\hat{x} \in \mathbb{R}^4 : |\hat{x}_2| \le 15^o, |\hat{x}_3| \le 4rad/s, |\hat{x}_4| \le 4rad/s\}$. In this case, we have 128 vertex models. Since the output matrix is constant, LMIs in Theorem 2 can be simplified if a constant Lyapunov matrix is chosen $\bar{P}_{e\mathbf{w}} = \bar{P}_e$; as before, the speed

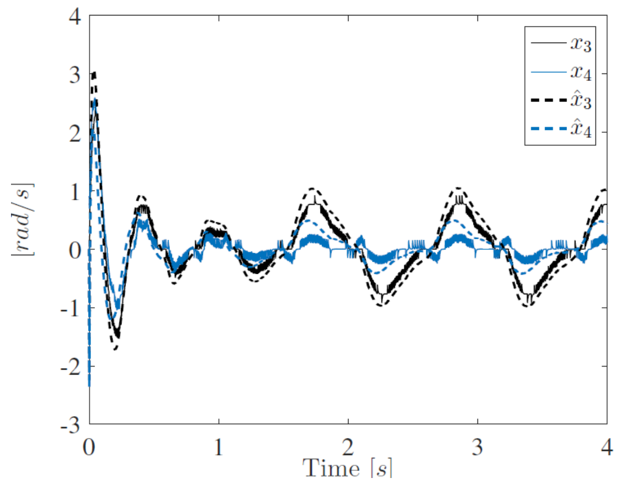

Fig. 7. Estimation of the velocities from the RIP in Example 3.

converge of the error towards zero is increase by means of the decay rate $\beta_o = 0.1$. Some of the computed gains are:

$$P_{1e} = \begin{bmatrix} 0.73 & 0.26 & -0.29 & 0.18 \\ 0.26 & 0.82 & 0.20 & -0.31 \\ -0.29 & 0.20 & 0.25 & -0.23 \\ 0.18 & -0.31 & -0.23 & 0.24 \end{bmatrix},$$

$$M_{1e1}=\begin{bmatrix} 0.49 & 0.08 & 0.41 & 0.35 \\ -0.04 & 0.53 & 0.18 & 0.30 \end{bmatrix}^T,$$

$$M_{1e9}=\begin{bmatrix} 00.51 & 0.03 & 0.39 & 0.31 \\ -0.00 & 0.53 & 0.21 & 0.35 \end{bmatrix}^T,$$

$$M_{2e3}=\begin{bmatrix} 0.11 & 0.08 & -0.10 & 0.03 \\ 0.06 & 0.11 & 0.00 & -0.12 \end{bmatrix}^T,$$

$$M_{2e32}=\begin{bmatrix} 0.14 & 0.12 & -0.12 & 0.02 \\ 0.03 & 0.07 & 0.00 & -0.14 \end{bmatrix}^T.$$

Real-time experiments have been conducted for $x(0) \approx [-\pi/10,\ -0.1,\ 0,\ 0]^T$ while for the observer $\hat{x}(0) \approx 0$, a quasi-LPV controller of the form (23) is computed with $x_a = [x_1,\ x_2,\ \hat{x}_3,\ \hat{x}_4]^T$ where $\hat{x}_3$ and $\hat{x}_4$ are obtained from the corresponding quasi-LPV observer. Fig. 7 plots how the observer estimates the velocities x_3 and x_4, while Fig. 8 shows the observation error. From Fig. 9 it can be seen that the positions of the vertical and horizontal arms are stabilized around the origin. Fig. 10 displays the control signal, it can be seen that thanks to the observer the signal is smooth as $\hat{x}_3$ and $\hat{x}_4$ are employed (the high-frequency and noisy signals x_3 and x_4 are not used in the controller). Additionally, the control signal is between the imposed bounds (see Remark 3), this helps prevent saturation in the actuators and possible mechanical wear.

V. Conclusion

A quasi-LPV dynamic output feedback controller for nonlinear descriptor systems has been presented. The approach is based on polytopic/convex modeling together with the direct Lyapunov method. Thanks to convexity, LMI conditions have been obtained. Moreover, performance indexes can be considered within the same LMI framework. Numerical examples and a real-time implementation in the rotatory inverted pendulum have been used to illustrate the advantage of our approach.

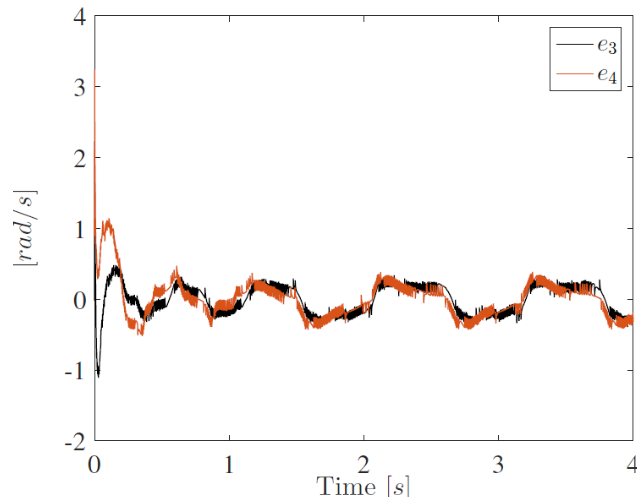

Fig. 8. Real-time estimation errors $e_3 = x_3 - \hat{x}_3$ and $e_4 = x_4 - \hat{x}_4$ from the RIP in Example 3.

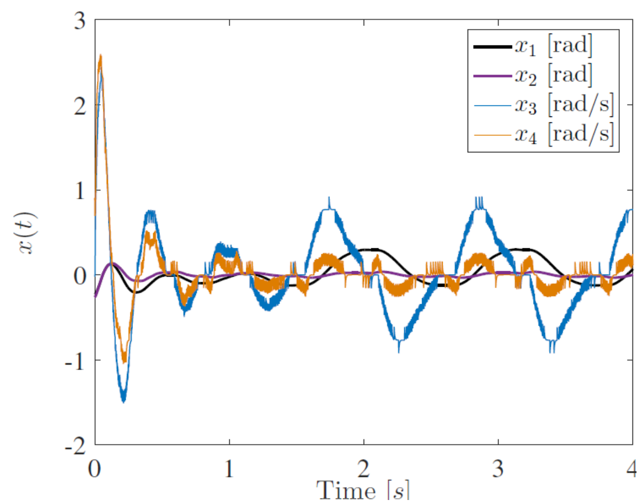

Fig. 9. Stabilization of the RIP in Example 3.

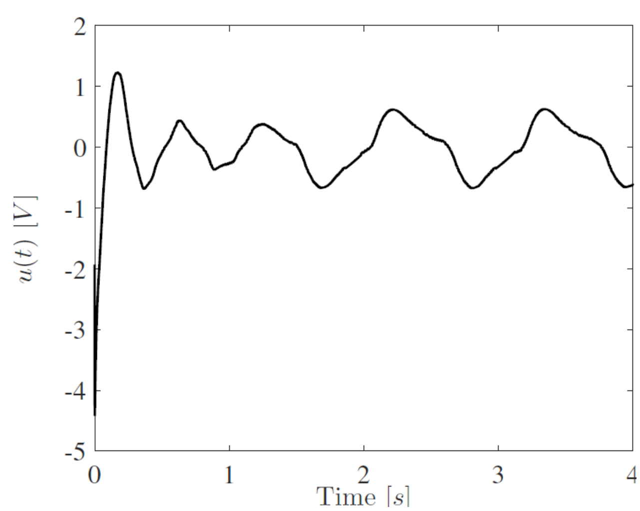

Fig. 10. Real-Time control signal applied to the RIP in Example 3.

Future work is oriented to develop robust controllers capable of dealing with parameter uncertainties and disturbances.

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Arturo Alvarado received his B. Sc. degree in mechatronics engineering from Universidad Politécnica de Pachuca, México, in 2022. He is currently pursuing the master degree in mechatronics. His research interests include control of nonlinear systems, output regulation theory, robotics, and real-time applications.

Tonatiuh Hernández-Cortés received his B.Sc. degree in robotics from the Escuela Superior de Ingeniería Mecánica y Eléctrica, Azcapotzalco Campus, Instituto Politécnico Nacional, Mexico City, Mexico, in 2005, and the M.Sc. and Ph.D. degrees in mechanical engineering from the Sección de Estudios de Posgrado e Investigación, Escuela Superior de Ingeniería Mecánica y Eléctrica, Zacatenco Campus, Instituto Politécnico Nacional, Mexico, in 2012 and 2016, respectively. He is currently a Full Professor in the Department of Control Engineering at the Universidad Autónoma del Estado de Hidalgo, Mexico. His research interests include nonlinear systems control, output regulation theory, robotics, fuzzy systems, and real-time applications.

Jaime González-Sierra received his B.Sc. degree in Mechatronics engineering from the Pachuca Polytechnic University (UPP), Hidalgo, Mexico, in 2008, his M.Sc. and Ph.D. degrees in Electrical Engineering from the Center for Research and Advanced Studies of the National Polytechnic Institute (CINVESTAV-IPN), Mexico, in 2010 and 2016, respectively. From 2016 to 2019, he belonged to the Conacyt young researchers program at the Technological Institute of La Laguna. Since 2022, he is a full-time professor at Unidad Profesional Interdisciplinaria de Ingeniería Campus Hidalgo, Instituto Politécnico Nacional, Mexico. His research areas are linear and nonlinear control of multi-agent systems and modeling of mobile robots. Dr. González-Sierra is part of the National Researcher System in Mexico at Level 1.

Víctor Estrada-Manzo received his Ph.D. degree in Automatic Control from the University of Valenciennes and Hainaut-Cambrésis, France, in 2015, on the subject of nonsingular convex descriptor systems. He worked as a Post-Doctoral fellowship at the Sonora Institute of Technology, Mexico from 2015 to 2018. He currently holds a Full Professor position at the Polytechnical University of Pachuca, Mexico. He is a member of the National Research System of Mexico since 2017. His research interests are analysis, design, and diagnosis of nonlinear control systems through descriptor quasi-LPV models and convex optimization techniques.